Quantum Integrability Towards Superconductivity

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1. Yang-Baxter Equation
2. RTT Relations

The Basis of Quantum Integrability

Yang-Baxter

$$R_{12}(\lambda, \mu)R_{13}(\lambda, \tau)R_{23}(\mu, \tau)$$
$$= R_{23}(\mu, \tau)R_{13}(\lambda, \tau)R_{12}(\lambda, \mu).$$

There are two contexts where we can physically interpret it:

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[a] One dimensional lattices like the Heisenberg spin chain.

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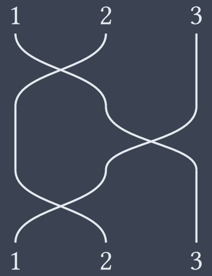
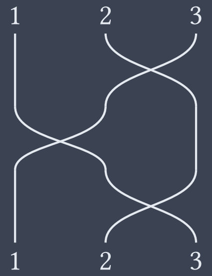
Braid Relation

Twisting the R matrix gets us a new but equivalent equation,

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23} \tag{1}$$

But what does it mean?

[a] One dimensional lattices like the Heisenberg spin chain.

$R_{12}R_{23}R_{12}$	$R_{23}R_{12}R_{23}$
	

CONSISTENCY!

The T Matrix & Conserved Quantities

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The Transfer Matrix

We define,

$$\tau(p) := \mathrm{Tr}_a\{T_a(p)\} \overbrace{\Longrightarrow}^{\text{It can be shown}} [\tau(p),\tau(k)] = 0$$

But why is this so useful?

Bethe Ansatz

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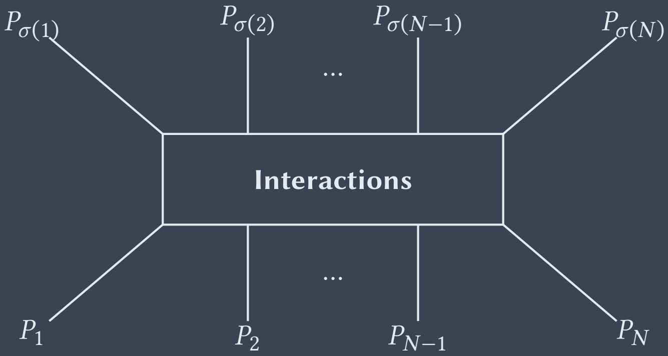
Factorised Scattering

The assumption of factorised scattering applied to a system is called the **Coordinate Bethe Ansatz**.

Factorised Scattering



Note

Here $\sigma \in \mathfrak{S}^N$ is some permutation since momentum conservation reflects a non-diffractive scattering process.

Algebraic Bethe Ansatz

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E.g. Hydrogen Atom.[4]

The Hubbard Model

Spectrum

- Applying the coordinate Bethe ansatz one sees that the charge and spin degrees of freedom separate
- The charge degrees of freedom called holons and spin degrees are called spinons.

Similarly we get the spectrum,

$$E = -2\sum_j \cos(k_j) + u(L-2N), \qquad P = \sum_j k_j \mod 2\pi$$

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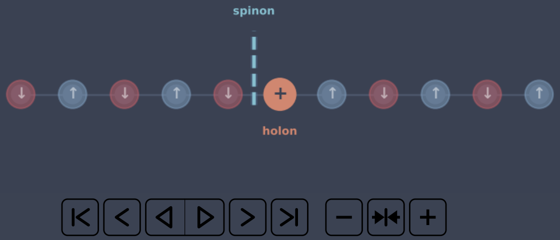

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Note here what happens with half-filling.

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Solution: Go to two dimensions - and give up integrability...

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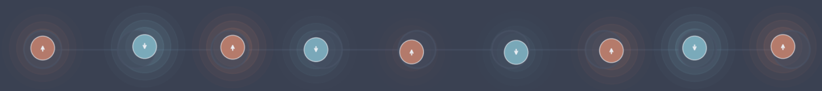

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Appendices

(Quantum) R matrix

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Remark: Often when describing the R matrix we do it in reference to two of three particles at a time e.g. R_{12}, R_{13} etc. The first one is easily represented by[1]

$$R_{12}(\lambda,\mu) = \underbrace{R(\lambda,\mu)}_{\mathscr{H}_1 \otimes \mathscr{H}_2} \otimes \underbrace{\mathbb{I}}_{\mathscr{H}_3} .$$

But the third and can't be written so easily - people write it in index form instead.

[1] As physicists.

Coordinate Bethe Ansatz

The most intuitive form[1] of this is called the *coordinate* Bethe Ansatz - let's understand it.

$$\Psi(\{q\}) = \sum_{\sigma,\tau} \mathscr{A}(\sigma \,|\, \tau) \exp\left\{ i \sum_{j=1}^{N} q_{\sigma(j)} p_{\tau(j)} \right\} \Theta\Big(q_{\sigma(1)} < q_{\sigma(2)} < \cdots < q_{\sigma(N-1)} < q_{\sigma(N)} \Big)$$