Quantum Integrability: An Investigation of the Bethe Ansatz and its Application to the Hubbard Model

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What is an integrable system?

Classical Integrability

- A classical dynamical system on a $2N$-dimensional phase space is **Liouville** integrable if it possesses N mutually commuting independent conserved charges, f_i - i.e. in involution with each other.
- The Arnold-Liouville theorem [1] guarantees the dynamics reduce to periodic motion on an N-torus, solvable by quadrature.
- Classically, integrability is a statement about the tractability of a system and solvability of its equations of motion.

Quantum Integrability

- No direct analogue of the Arnold-Liouville theorem exists in quantum mechanics; it is replaced by a family of pairwise commuting conserved charges, I_k, one of which is the Hamiltonian. [2]
- The quantum inverse scattering method fills this gap, providing a framework centred on quantum groups and the Yang-Baxter equation through which quantum integrable systems may be solved.

Quantum Integrability

One may then ask, **what is a quantum group?** An excellent question - both physically and mathematically[4].

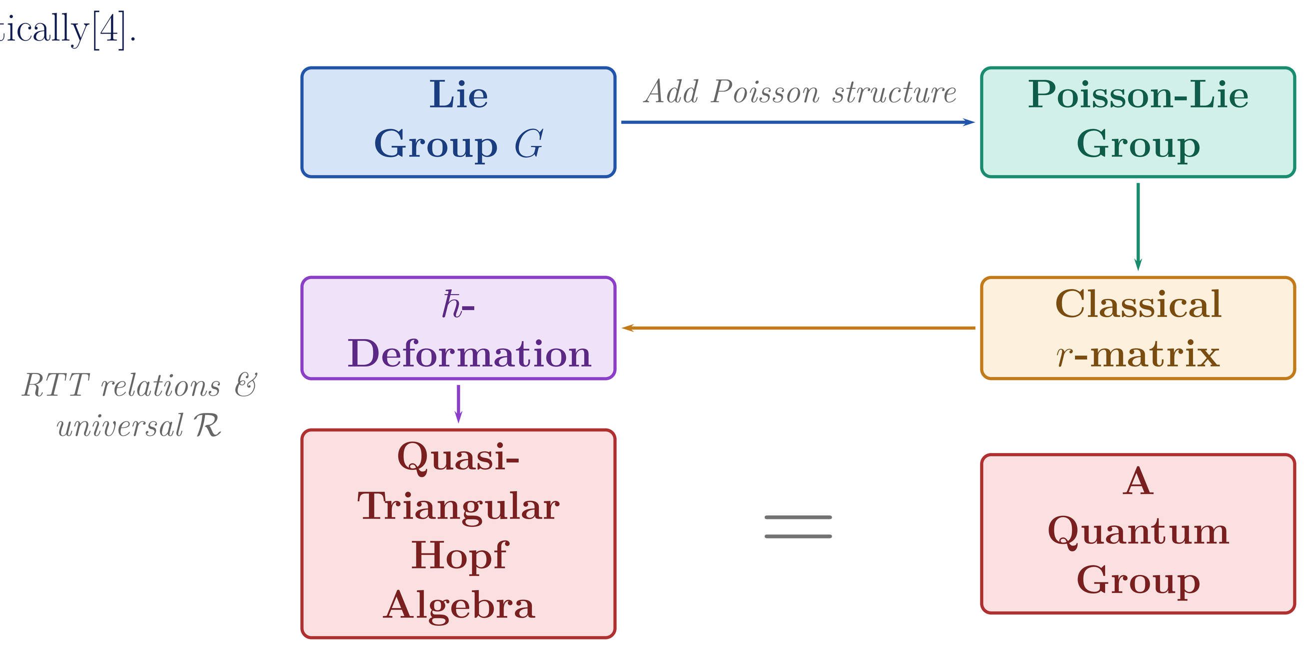

Figure: The mathematical roadmap describing how we construct a quantum group.

Quantum Groups underpin all of quantum integrability. They provide the mathematical description of quantum systems in a very subtle but interesting way. There are two relations which define a quantum group:

1. The **Yang-Baxter** equation.
2. The *RTT* Relations.

Both use the central object of an R-matrix.

- *Mathematically*, the R-matrix is a solution of the Yang-Baxter equation as seen below.
- *Physically*, it describes a scattering process between two particles with momenta given by its argument.
- These numbers are called spectral parameters.
- We're most interested when this represents swapping particles.

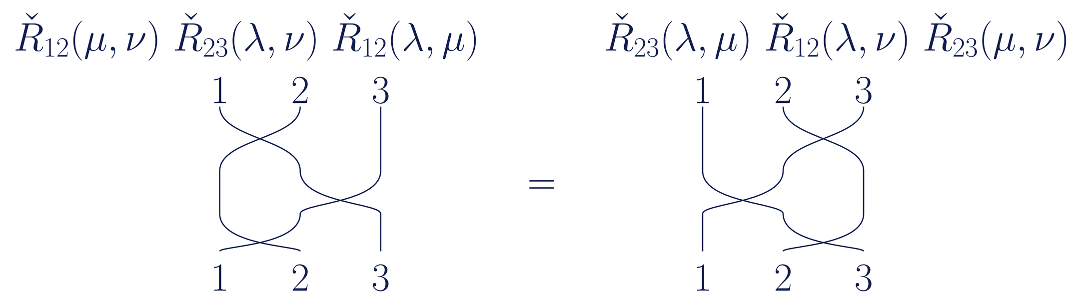

Figure: A graphical representation of the Yang-Baxter equation. Technically, this is the Braid Relation but it is equivalent to the usual Yang-Baxter. This can be seen through the twisted R-matrix, $\check{R}_{ab} = P_{ab}R_{ab}$, where P is a permutation operator.

The second, more physically relevant equations are the RTT Relations:

$$R_{ab}(\lambda,\mu)\,T_a(\lambda)\,T_b(\mu) = T_b(\mu)\,T_a(\lambda)\,R_{ab}(\lambda,\mu)$$

- The solution $T_a(\lambda) = \prod_{k=1}^{N} R_{ka}(p_k,\lambda)$ admits a natural interpretation: the **monodromy matrix** T encodes the overall effect of scattering a fictitious auxiliary particle a off the entire system.
- The physically important object is the **transfer matrix**, $\tau(\mu) \coloneqq \mathrm{Tr}_a[T_a(\mu)]$. Via the RTT relations, transfer matrices commute for different spectral parameters.
- Expanding $\tau(\mu)$ in μ generates an infinite tower of mutually commuting charges - provided one of them is the Hamiltonian.

The Bethe Ansatz

The term **Bethe Ansatz** has acquired many different meanings throughout the past century. The two we will focus on are the:

- Coordinate Bethe Ansatz
- Algebraic Bethe Ansatz

Coordinate Bethe Ansatz: We propose a wave function of the form,

$$\Psi(q_1,q_2,\ldots,q_N) = \sum_{\tau\in\mathfrak{S}_N} \mathcal{A}(\tau)e^{i\sum_i q_i p_{\tau(i)}}.$$

This solution is a superposition of plane waves whose amplitudes are determined by demanding consistency with boundary conditions and two-body scattering. We can reduce this solution to just N coupled algebraic equations for the quasi-momenta $\{k_j\}$ - the **Bethe Equations**.

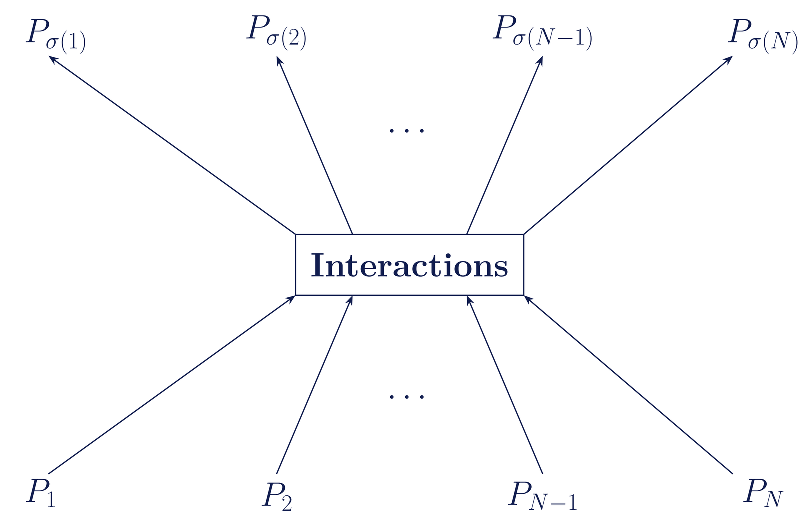

Figure: The process of factorised scattering i.e. the process of all events occuring in 2-to-2 events - forms the bedrock of the Coordinate Bethe Ansatz. This leads to the conclusion that the outgoing momenta are a permutation of the incoming momenta.

Algebraic Bethe Ansatz: Otherwise known as the Quantum Inverse Scattering Method, here we recast the construction in the language of quantum groups. Two of the elements of the monodromy matrix are identified as creation and annihilation operators acting on a pseudo-vacuum $|\Omega\rangle$. States are then built out of concatenation of the creation operator, $B(\lambda)$, as

$$|\lambda_1,\ldots,\lambda_N\rangle = \prod_{i=1}^{N} B(\lambda_i)\,|\Omega\rangle\,.$$

The **Bethe Equations** emerge from this language when we require these to be eigenstates of the transfer matrix and hence obeying all conservation laws of the system.

The Hubbard Model

The One Dimensional Hubbard model [5] describes electrons hopping on a chain with on-site Coulombic interaction,

$$H = -\sum_{i,\sigma} c^\dagger_{i,\sigma}c_{i-1,\sigma} + c^\dagger_{i-1,\sigma}c_{i,\sigma} + u\sum_j n_{j,\uparrow}n_{j,\downarrow}.$$

Despite its apparent simplicity, the model is extremely physically rich. It captures narrow-band electronic behaviour, the **Mott insulating** phase transition, and more! Notably, it is thought to be integrable in only one spatial dimension.

- The Hubbard model's integrability is derived via Shastry's R-matrix — notably not of difference form, i.e. $R(\lambda,\mu) \neq R(\lambda-\mu)$, making it a rare exception.
- Following the standard procedure yields the **Lieb-Wu equations**, which describe the dynamics of the system.
- These naturally separate into a **charge** and **spin** sector, reflecting the existence of two quasi-particles: the **holon** and the **spinon**.

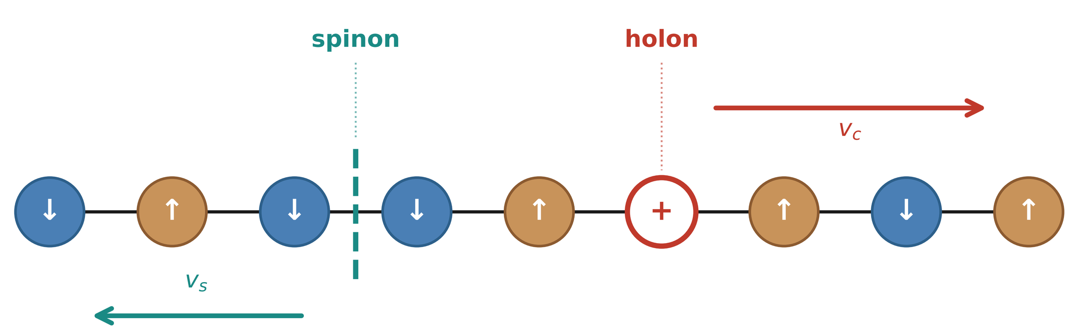

Figure: A schematic showcasing the decoupling of spin and charge degrees of freedom called the **spinon** and the **holon** respectively.

Further Work

To discuss one particular aspect of novel further work here, we first must understand the Calogero-Moser-Sutherland model. The CMS model, strictly speaking, a family of integrable models rather than a single one. In its simplest - the rational - incarnation, it describes N particles confined to a line interacting via a pairwise inverse-square potential.

$$H = -\frac{1}{2}\sum_{i=1}^{N}\partial_i^2 + \frac{g}{2}\sum_{i\neq j}\frac{1}{|q_i-q_j|^2}.$$

Replacing the rational potential with a sine or hyperbolic sine interaction yields the trigonometric and hyperbolic models with the most general member being expressed using the Weierstrauss $\wp$-functions - the elliptical model. It has been shown [3] that the first three mentioned can arise from the quantum Hall system on varying topologies in the Lowest Landau Level:

CMS Model		Quantum Hall Topology
Rational	$\Longleftrightarrow$	Disc
Trigonometric	$\Longleftrightarrow$	Cylinder
Hyperbolic	$\Longleftrightarrow$	Cylinder[a]
Elliptical	$\overset{?}{\Longleftrightarrow}$	Torus

This final identification is seemingly novel and will be further explored.

[a] Coupled with a Fourier Transform.

Conclusion

- This work traces a path from classical Liouville integrability to the exact solution of the one-dimensional Hubbard model via the Bethe Ansatz.
- The Yang-Baxter equation provides the algebraic backbone from which commuting conserved charges, and hence integrability, are constructed.
- Shastry's R-matrix places the Hubbard model on rigorous algebraic footing, with the resulting Lieb-Wu equations yielding non-perturbative exact results e.g. Mott insulating phase and spin-charge separation.
- Both lie entirely beyond the reach of conventional perturbation theory.
- Much has been left aside - the Thermodynamic Bethe Ansatz, correlation functions, and the relation to superconductivity each constitute substantial programmes in their own right.

References

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