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Quantum Integrability: An Investigation of the Bethe Ansatz and its Application to the Hubbard Model

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Symmetry dictates interaction.

— C.N. Yang

Abstract

This thesis investigates quantum integrability through the lens of the Bethe Ansatz and its application to the one-dimensional Hubbard model. The classical foundations are developed as motivation for the quantum Yang-Baxter equation, which is approached from the perspectives of the Heisenberg spin chain and factorised scattering theory.

The Calogero-Moser-Sutherland family of models is studied and its spectrum obtained, with a known duality to the quantum Hall effect on varying geometries reviewed and extended: a conjecture is proposed that placing the quantum Hall effect on a torus yields the elliptic Calogero-Sutherland model upon projection to the Lowest Landau Level.

The Hubbard model is solved via the coordinate Bethe Ansatz, its hidden $SO(4)$ symmetry uncovered, and the algebraic Bethe Ansatz constructed using graded vector spaces and Shastry's R-matrix, recovering the Lieb-Wu equations from an algebraic perspective.

Keywords: Quantum Integrability; Bethe Ansatz; Hubbard Model; Yang-Baxter Equation; Calogero-Moser-Sutherland Models; Quantum Hall Effect

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1. Introduction

Integrability is a difficult term to adequately define. In its most intuitive, although informal, form it can be defined as *the application of varying methods to utilise conserved quantities in pursuit of the analytic solutions relevant to physical systems.* This is often stated to in turn define an integrable system. In a slightly more formal setting we may define the same as follows.

Definition 1 (Integrability)**.** *The study of methods concerning the dimensional reduction of the phase space manifold for which a dynamical system exists on. Here a dynamical system refers to a system where the relevant quantities, e.g. the positions of particles subject to a potential, the quantum mechanical wavefunction, etc. are functions of time. Equivalently, it is the separation of a composite system into a number of non-interacting subsystems.*

Before diving a bit further into this topic - we should cover some brief background. Many readers may have already come across this material from the point of view of analytic mechanics. However, this may not be sufficient due to a lack of mathematical sophistication. For a more thorough introduction to both the physical and mathematical aspects we recommend [1]. It is for these reasons that we shall first explore a non-exhaustive but detailed background. The background will also serve to unify notation.

With this background established, we then develop the mathematical sophistication necessary to discuss symplectic geometry and Hamiltonian reduction - the coadjoint orbit, the momentum map, and the Marsden-Weinstein theorem, following the text [2]. These are not merely formal embellishments. The structures uncovered here resurface, often in a strikingly direct fashion, in the quantum theory, and it is for this reason that we afford them the attention we do.

From there, we turn to the central subject of the text. The quantisation of the classical picture leads us naturally to quantum groups and the quantum Yang-Baxter equation - the organising principle of quantum integrability. We motivate this object from two directions, namely the Heisenberg spin chain and the theory of factorised scattering, before arriving at the Bethe Ansatz as a practical tool for solving quantum integrable systems. Along the way, we study the Calogero-Moser-Sutherland family of models, whose spectrum reveals a beautiful and perhaps surprising connection to the quantum Hall effect - one that serves as both a recurring thread throughout the text and the basis for novel conjectural work in the final chapter based on [3].

The crescendo of the thesis is the Hubbard model, a lattice model of considerable importance to condensed matter physics and one of the most celebrated examples of a quantum integrable system. We work through this based on [4]. We solve it via the coordinate Bethe Ansatz, uncover its hidden $SO(4)$ symmetry, and revisit it through the more powerful algebraic framework - requiring the formalism of graded spaces and Shastry's R-matrix. It is here that the full power of the machinery built throughout the text is brought to bear.

2. Classical Integrability

2.1. Classical Mechanics

2.1.1. Classical Dynamical Systems

For completeness, we shall begin from Hamilton's principle of stationary action and further derive all we will need.

Definition 2. *The classical action functional of a system of N particles, where summation over repeated indices is implied, is given by*

$$S[q] = \int_{t_0}^{t_1} dt\, \mathcal{L}(q, \dot{q}, t), \tag{2.1}$$

where the integrand consists of the ***Lagrangian*** *which is simply the difference between kinetic and potential energy i.e. $\mathcal{L} = T - V$. An alternative formulation of the action can be expressed via the* ***Hamiltonian****. Which is defined via Legendre transform as follows*

$$\mathcal{H} = p_i \dot{q}^i - \mathcal{L}, \quad S = \int_{q^j(t_0)}^{q^j(t_1)} p_i\, dq^i - \int_{t_0}^{t_1} dt\, \mathcal{H}. \tag{2.2}$$

This object is used throughout modern theoretical physics and is of supreme importance. For example,

Axiom 1 (Hamilton's Principle of Stationary Action)**.** *Hamilton's principle states that along physical trajectories taken by classical systems the variation of the action is null i.e.*

$$\delta S = 0.$$

Utilising the calculus of variations we can derive the equations of motion (EoM) in terms of the Lagrangian or Hamiltonian equally. We will find that the Hamiltonian formalism is far more useful for our needs in the context of integrability and therefore we will confine ourselves exclusively to the Hamiltonian formalism. Using Hamilton's principle[1] we find Hamilton's equations,

$$\dot{q}^i = \frac{\partial \mathcal{H}}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q^i}. \tag{2.3}$$

We briefly note for completeness that pursuing the same argument in the Lagrangian formalism then we find the EoM in the form of the Euler-Lagrange Equations

$$\frac{\mathrm{d}}{\mathrm{dt}} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \frac{\partial \mathcal{L}}{\partial q^i} = 0. \tag{2.4}$$

[1]For systems with conserved energy it is also known as Maupertuis' principle.

This is a basic form of the EoM but we can rearrange it into a far superior form. This is what we will call a *Poisson Bracket* in its most simple form. It will be "defined" as

$$\{f, g\}_{\text{PB}} \equiv \{f, g\} \coloneqq \frac{\partial f}{\partial p_{[i}} \frac{\partial g}{\partial q^{i]}}. \tag{2.5}$$

There is slight niche point here. Technically, when we have defined a Poisson Bracket (PB) in this fashion, we have already assumed a specific form it must take. This will prove suboptimal for a generality standpoint. In particular, this assumption is disastrous when applied to constrained systems - a point that will be discussed later. For full generality, we shall define a PB not by its form but by its properties.

Definition 3 (Poisson Bracket)**.** *A Poisson Bracket as defined on the space of functions over a manifold $\mathscr{P}$ denoted as $\mathscr{F}(\mathscr{P})$ is a bilinear, skew-symmetric map*

$$\mathscr{F}(\mathscr{P}) \times \mathscr{F}(\mathscr{P}) \to \mathscr{F}(\mathscr{P}),$$

which abides by the Jacobi identity and the Leibniz product rule.

The first three properties also classifies the PB as a Lie Bracket and therefore the function space a Lie algebra. In discussing the PB as a way to obtain the EoM, we must first make particular reference to the manifold $\mathscr{P}$ for which the function space is defined over. This is the *phase space* of the system. Properly defined, it is the cotangent bundle of the configuration space i.e. the space defined by the generalised coordinates q^i. In more layman's terms, we think of the configuration space and at each point classify a space of momentum covectors p_i. Therefore, to know the exact state the system is in we must know the positions and momenta which is precisely the location on the phase space manifold.

Now that we have properly defined our phase space we can see how our EoM fall out far more simply when rewritten in this language. In the case of a system with N generalised coordinates the phase space is a $2N$-dimensional orientable manifold[2] and we take an element $x \in \mathscr{P}$. Defining the gradient over phase space as the concatenation of the gradient in position space and momentum space we can write the EoM as

$$\dot{x} = -J \cdot \boldsymbol{\nabla}\mathcal{H},$$

for some anti-symmetric rank two tensor J. Taking inspiration from our original "definition" of the PB we define it here as

$$\{f, g\}(x) = J^{ij}(\partial_i f)(\partial_j g).$$

Which lets us describe the time evolution in the following way

$$\dot{x} = \{\mathcal{H}, x\}.$$

Furthermore, since the PB is defined to obey Leibniz' rule then for any $f \in \mathscr{F}(\mathscr{P})$ we have that

$$\dot{f} = \frac{\partial f}{\partial x^i}\dot{x}^i = \frac{\partial f}{\partial x^i}\{\mathcal{H}, x^i\}.$$

[2]See Prof. Frolov's Differential Geometry problems for a proof.

This coupled with the skew-symmetry of the PB tells us that Hamiltonian itself is conserved. This is the first taste of Nöther's Theorem and actually alludes to an assumption we've made without mentioning - that these functions over phase space have no *explicit* time dependence. Thereby telling us that if and only if the Hamiltonian has no time dependence, hence the system has time translation invariance, then the Hamiltonian or energy is conserved.

In the interest of generality, we can repeat this analysis for any manifold disregarding the phase space aspect. If there exists a PB structure on the manifold then it is said to be a Poisson manifold. Further, if J is invertible everywhere on the manifold then it is called a symplectic manifold. One can then inquire about the dimensionality of the manifold. In general the rank of J is less than or equal to the dimension of the space, although this can vary along the manifold itself. However, since J is anti-symmetric then it can only be invertible if it exists in an even dimensional space. Therefore, a symplectic manifold necessarily must be of even dimension. In this case, the PB is called non-degenerate and we can define the symplectic two form ω such that $\omega_{ik}J^{kj} = \delta_i^j$. The anti-symmetry of ω and the Jacobi identity of J yields the fact that $d\omega = 0$.

Given a Poisson manifold $\mathscr{P}$ and any function $f \in \mathscr{F}(\mathscr{P})$ we can define the Hamiltonian vector field $\xi_f \coloneqq \{f, \cdot\}$. The components of which are $\xi_f^i = J^{ji}\partial_j f$ and hence $\partial_j f = \omega_{ji}\xi_f^j$. The correspondence between the symplectic form and the PB can be expressed simply as

$$\omega(\xi_f, \xi_g) = \{f, g\} = \xi_f g = -\xi_g f. \tag{2.6}$$

In the language of Lie algebras, we define a Casimir function $C \in \mathscr{F}(\mathscr{P})$ as a function which Poisson commutes with every other function. That is to say, it is annihilated by every Hamiltonian vector field. Furthermore, by the above relation ξ_C is trivially null. Borrowing now on our understanding from Quantum Mechanics, we have a set of Casimir operators and wish to further study a given system. We therefore classify our state space using what would be called "good quantum numbers". Here this translates to defining a level set $\mathscr{P}_C$ where a set of Casimir functions $C_i(x)$ are all set to constants C_i. Then any Hamiltonian vector field is tangent to the level set since $\xi_f C_i = 0$.

Finally, given a certain symplectic manifold how may we change it to find equivalent spaces? That is to say, consider a smooth coordinate transformation $x \to \hat{x}(x)$. Let's investigate then how Hamilton's equations (2.1.1) transform

$$\frac{\mathrm{d}\hat{x}^i}{\mathrm{dt}} = \frac{\partial \hat{x}^i}{\partial x^j}\frac{\mathrm{d}x^j}{\mathrm{dt}} = -\frac{\partial \hat{x}^i}{\partial x^j}J^{jk}(x)\frac{\partial}{\partial x^k}\mathcal{H}(x) = -\underbrace{\frac{\partial \hat{x}^i}{\partial x^j}\frac{\partial \hat{x}^k}{\partial x^l}J^{jl}}_{=\hat{J}^{ik}(\hat{x})}\frac{\partial}{\partial \hat{x}^k}\mathcal{H}. \tag{2.7}$$

This also tells us that the Hamiltonian is left unchanged under this change of coordinates. These diffeomorphisms are called **canonical transformations** and the covariance of J implies the covariance of the PB itself. Furthermore, for infinitesimal transformations $\hat{x}^i = x^i + \xi^i$ one can show via the Jacobi identity that for a Hamiltonian vector field the Lie derivative $\mathscr{L}_\xi J = 0$ and hence for symplectic manifolds where ω is well defined everywhere we also have that $\mathscr{L}_\xi \omega = 0$. Which is why these transformations are also known as **symplectomorphisms**.

Now that we've built up a sufficient understanding to converse in the language of classical dynamical systems, we can begin to move on to the concept of Liouville integrability. The central statement here i.e. what we will endeavour to show and then to use, is that there exists a class of systems for which solutions of Hamilton's equations can always be found by quadrature. Dynamical systems endowed with this property are labelled **Liouville integrable** or completely integrable. This property succinctly put takes the form of the Arnold-Louiville Theorem. A more formal presentation of the theorem and an outline of the proof is discussed in Appendix A.2. The notion of Liouville's theorem is central to integrability and hence any unfamiliar readers are recommended to read through the appendix to gain some familiarity with terminology and concepts before moving onto 2.2.

What we have accomplished so far has been quite general in nature. A rational question to ask now would be that of embedding our notion of integrability with physical intuition. We do this by analysing the action and equations of motion of a general system. In particular, any symmetries they may have - called in the literature *variational symmetries*. One can then use this language to find a method of finding necessary conditions on integrability via Nöther's theorem. This fundamental part of integrability is luckily often covered in elementary analytical mechanics. For the lay reader who may not be familiar with the formalisation of Nöther's theorem and its proof or are in need of a refresher, they should consult A.1.

2.1.2. Lax Representation

This is most likely the first topic new to those readers with a basic understanding of analytical mechanics. The modern literature uses a much more formulaic technique to investigating the integrability of dynamical systems, the notion of *Lax pairs*.

Definition 4 (Lax Representation)**.** *If there exist two matrix valued functions of the phase space of a dynamical system called a Lax pair, L and M such that*

$$\dot{L} = [M, L]. \tag{2.8}$$

then the system is said to admit a Lax representation.

The natural question, of course, is if we can solve for L. We will do this explicitly by invoking the adjoint operators

$$\mathrm{ad}_X \star := [X, \star], \qquad \mathrm{Ad}_X \star := X \star X^{-1}.$$

The defining relationship of the Lax pair can then be written as

$$\dot{L} = \mathrm{ad}_M L.$$

Since M and L are both functions of time, in general, then we solve this separable ODE as

$$\begin{aligned} \mathrm{d}L \cdot L^{-1} &= \mathrm{d}t \, \mathrm{ad}_M = \mathrm{d} \ln(L) \,, \\ \ln(L(t)) - \ln(L(0)) &= \int_0^t \mathrm{d}\tau \, \mathrm{ad}_{M(\tau)}, \\ L(t) &= \exp\left\{ \int_0^t \mathrm{d}\tau \, \mathrm{ad}_{M(\tau)} \right\} L(0). \end{aligned}$$

Now we use the linearity of the adjoint operator to bring the integral into matrix and the Campbell identity [5]

$$e^{\mathrm{ad}\,X} = \mathrm{Ad}_{e^x}\,. \tag{2.9}$$

Thereby rewriting the solution to L as

$$L(t) = \mathrm{Ad}_{\int_0^t \mathrm{d}\tau M}\, L(0).$$

For ease of notation, however, we define the following matrix valued function g and hence can rewrite the solution in the standard presentation

$$\begin{aligned} g(t) := \exp\left\{\int_0^t \mathrm{d}\tau\right\} M(\tau) \implies M(t) = \dot{g}g^{-1} \\ L(t) = g(t)L(0)g(t)^{-1}. \end{aligned} \tag{2.10}$$

It's vital to note here that these solutions are not unique and can be gauge transformed as follows

$$L \to hLh^{-1}, \qquad M \to hMh^{-1} + \dot{h}h^{-1}. \tag{2.11}$$

The reader should note the similarity of this construction to the Von Neumann equation [6] in quantum mechanics

$$i\partial_t \rho = \mathrm{ad}_H\, \rho,$$

where ρ is the density matrix, H the Hamiltonian operator, and we've set $\hbar = 1$. It should be obvious then that the eigenvalues of L are constant since they are analogous to the probabilities of being in a certain eigenstate. Here and throughout the rest of this work we will use the following notation,

$$L_1 = L \otimes \mathbb{1} = \sum_{ij} L_{ij}\, E_{ij} \otimes \mathbb{1},$$

where E_{ij} are the canonical basis matrices i.e. $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Similarly,

$$L_2 = \mathbb{1} \otimes L, \quad T_{12} = \sum_{ij,kl} T_{ij,kl}\, E_{ij} \otimes E_{kl}, \quad T_{21} = \sum_{ij,kl} T_{ij,kl} E_{kl} \otimes E_{ij}.$$

In general, we say that L_α is the embedding of L into q copies of our space

$$L_\alpha := \underbrace{\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes L}_{\alpha} \otimes \mathbb{1} \otimes \cdots$$

and similarly for $T_{\alpha\beta}$ in the α and β positions. We must also define two more properties of this language to continue our discussion. Firstly, the partial trace of an operator

$$\mathrm{Tr}_\alpha\, X := \sum_{i_1 j_1, i_2 j_2, \ldots} X_{i_1 j_1, \ldots i_q j_q}\, \mathrm{Tr}(E_{i_\alpha j_\alpha}) E_{i_1 j_1} \otimes E_{i_2 j_2} \otimes \cdots \hat{E}_{i_\alpha j_\alpha} \otimes \cdots E_{i_q j_q} \tag{2.12}$$

where the hat indicates the omission of the α position matrix. Secondly, the PB of two operators is defined in the obvious way

$$\{L_1, L_2\} := \sum_{ij,kl} \{L_{ij}, L_{kl}\} E_{ij} \otimes E_{kl}.$$

Theorem 1 (Babelon-Viallet). *The involutive property of the eigenvalues of the Lax operator L is equivalent to the existence of a function r_{12} on the phase space such that*

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]. \tag{2.13}$$

Such a function is called the r-matrix. In the case where r is a function of the system's dynamical variables then it is called the dynamical r-matrix.

We will also note that the r-matrix is clearly antisymmetric.

Corollary 1 (Classical Yang-Baxter Equation). *Seeing such a clean relation between the Poisson bracket of two Lax matrices, one might ask what happens if we were to utilise the Jacobi identity as defined for Poisson brackets? This precisely yields,*

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0. \tag{2.14}$$

This construction is beautiful - but possesses a potentially detrimental flaw. As is discussed more thoroughly in A.3, the quantity $I_k := \operatorname{tr} L^k$ is an integral of motion with the usual independence criterion of Poisson commutativity. However, this rather naturally leaves the number of independent integrals of motion obtainable in this fashion capped at the rank of the Lax matrix. Which is of particular issue for systems with a high dimensionality compared to the rank of the Lax matrix. This problem is fixed via the introduction of a quantity known as the **spectral parameter**. This does not have to be, and is in general not, a physical quantity. Indeed, it is oft just some $\lambda \in \mathbb{C}$. We do necessitate that the Lax pair (L, M) are both functions of this parameter and the defining commutation relation holds for all values of λ. Indeed, as shown in [2], if one further assumes that the pair are rational functions of the spectral parameter then the Lax matrix can be diagonalised around its poles. Thereby telling us that the coefficients of the Laurent expansion are integrals motion thus increasing the number of independent integrals of motion and fixing the aforementioned issue. The procedure outlined in the text for building integral models and the results of which we have used here is known as the *Zakharov-Shabat* construction.

The natural question now is that given this new spectral parameter framework, can we apply the same logic as in Corollary 1 ? The answer is yes. Doing so we also assume that the r-matrix will be a function of the two relevant spectral parameters. In particular, a function of the difference between the two spectral parameters with the same anti-symmetric property that we discussed above. This notion behind why the r-matrix being a function of the difference between spectral parameters features again and is seemingly related to a sort of translational invariance in parameter space. This makes sense since the only physical quantities seem to be related to the expansion around the different poles of L. So if we translate the entire complex plane then the nature of the expansions themselves won't change much. This process, as outlined in [7], results in

$$[r_{12}(\lambda_1 - \lambda_2), r_{13}(\lambda_1 - \lambda_3)] + [r_{12}(\lambda_1 - \lambda_2), r_{23}(\lambda_2 - \lambda_3)] + [r_{13}(\lambda_1 - \lambda_3), r_{23}(\lambda_2 - \lambda_3)] = 0. \tag{2.15}$$

Thus concludes our preliminary discussion of classical integrability. Now that we have a sufficient physical motivation we can move forward to develop a more sophisticated mathematical understanding of the problem at hand in the form of symplectic geometry. It has been a purposeful choice to forego examples of the use cases of Liouville integrability and the Lax Representation for brevity. Many examples can be found in detail throughout the literature but we recommend [1], [2].

2.2. Symplectic Geometry & Group Theory

In this section, we explore the mathematical background motivating our entire view of integrability going forward. We begin by discussing the coadjoint orbit of a lie group and how it ties in with integrability via Hamiltonian reduction. We conclude with an example being the cotangent bundle of a Lie group.

2.2.1. Coadjoint Orbits of a Lie Group

For our further discussion, let G be a connected Lie group and $\mathfrak{g}$ its Lie algebra. Denote $\mathfrak{g}^*$ the dual of $\mathfrak{g}$ i.e. the space of linear continuous maps $\mathfrak{g} \to \mathbb{R}$. We can define the gradient of smooth functions over $\mathfrak{g}^*$, the space $\mathscr{F}(\mathfrak{g}^*)$ as

$$\langle m, \nabla f(l) \rangle := \lim_{t\to 0} \frac{f(l+tm) - f(l)}{t}, \qquad l, m \in \mathfrak{g}^*.$$

Definition 5 (Kirillov-Kostant Bracket)**.** *The KK Bracket is defined as*

$$\{f, h\} := \langle l, [\nabla f(l), \nabla h(l)] \rangle . \tag{2.16}$$

Thereby, it endows the dual space with the structure of a Poisson manifold. Fixing a basis of the algebra e_i such that

$$[e_i, e_j] = c_{ij}^k e_k$$

defines also the basis of the dual space by $\langle e^i, e_j \rangle = \delta^i_j$. In these coordinates we write $\mathfrak{g}^* \ni l = l_i e^i$ and hence

$$\nabla f := e_i \frac{\partial f}{\partial l_i} \implies \nabla l_j = e_j.$$

The KK bracket between these coordinates, then, is

$$\{l_i, l_j\} = \langle l, [e_i, e_j] \rangle = c_{ij}^k \langle l, e_k \rangle = c_{ij}^k l_k. \tag{2.17}$$

One might notice that at $l = 0$ the KK bracket is entirely degenerate. This may seem troublesome but is actually symptomatic of a particularly beautiful point. To understand this, we first recover some terminology from 2.1.1.

Definition 6 (Symplectic Leaves)**.** *Suppose there exists a complete set of m functions $C_i \in \mathscr{F}(\mathscr{P})$ is a Casimir function. By definition, then, the Hamiltonian vector field generated by it, ξ_{C_i}, vanishes. Then the one-form dC_i is in the kernel of the operator J since $\xi_{C_i} = J\, \mathrm{d}C_i$. For non-constant Casimir this means that the rank of J is less than dimension of the phase space. That is to say, the PB is degenerate.*

Recall the definition of a level set $\mathscr{P}_c$ as in (A.4). Then we trivially have that the tangent space of this level set is just what is not in $\ker J$ at any point i.e. $T_x\mathscr{P}_c = \operatorname{Im} J(x)$. Frobenius' Theorem 6 *tells us that the level set is an integral submanifold of $\mathscr{P}$ and the tangent space at any point being the kernel of J tells us that ω is non-degenerate. By theorem 13 in [8] we have*

$$\mathrm{d}\omega \propto \xi_f \cdot \omega(\xi_g, \xi_h) - \xi_g \omega(\xi_f, \xi_h) + \xi_h \omega(\xi_f, \xi_g) - \omega\left([\xi_f, \xi_g], \xi_h\right) + \omega\left([\xi_f . \xi_h], \xi_g\right) - \omega\left([\xi_g, \xi_h], \xi_f\right).$$

Then, by the linearity and skew symmetry of ω we have

$$\mathrm{d}\omega \propto \xi_f \cdot \omega(\xi_g, \xi_h) + \xi_g \omega(\xi_h, \xi_f) + \xi_h \omega(\xi_f, \xi_g) - \omega\Big(\cancelto{0}{[\xi_f + \xi_g + \xi_h, \xi_f + \xi_g + \xi_h]},\ \xi_f + \xi_g + \xi_h\Big)$$

and expressing now the forms in terms of the PB we get

$$\mathrm{d}\omega\left(\xi_f, \xi_g, \xi_h\right) = \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}$$

which vanishes due to the Jacobi identity. Therefore $\mathscr{P}_c$ is a symplectic manifold. This tells us that the phase space foliates into integral even-dimensional submanifolds called symplectic leaves with symplectic form induced by the phase space in its entirety.

Following this, we need to define the titular object.

Definition 7 (Coadjoint Orbit)**.** *Recall the definitions of the adjoint action of a Lie group and algebra respectively,*

$$\mathrm{Ad}_g X := gXg^{-1}, \qquad \mathrm{ad}_X Y := [X, Y], \qquad g \in G;\ X, Y \in \mathfrak{g}. \tag{2.18}$$

The coadjoint action of G in the dual space is defined as

$$\mathrm{Ad}^*_g l(X) := l(\mathrm{Ad}_{g^{-1}} X) = l(g^{-1} X g). \tag{2.19}$$

Similarly, the coadjoint action of the algebra on the dual space is defined via the derivative map such that

$$\mathrm{ad}^*_X l(Y) := -l(\mathrm{ad}_X Y) = -l([X, Y]). \tag{2.20}$$

Under this coadjoint action we can split the dual space into orbits. The definition of an orbit passing through a point $m \in \mathfrak{g}^*$ is

$$\mathscr{O}_m := \{\mathrm{Ad}^*_g m \,|\, g \in G\}. \tag{2.21}$$

Likewise, we also define the stabiliser group of this point G_m and hence the orbit can be expressed as

$$\mathscr{O}_m \simeq G/G_m.$$

It can also be shown that the tangent space to the orbit at m can be identified as

$$T_m \mathscr{O}_m \simeq \mathfrak{g}/\mathfrak{g}_m, \qquad \mathfrak{g}_m := \{X \in g \,|\, \mathrm{ad}^*_X g = 0\}.$$

Following our steps from earlier it is natural to define the action of a vector field generated by some $X \in \mathfrak{g}$ on a function $f \in \mathscr{F}(\mathfrak{g}^*)$

$$\xi_X f(l) := -\left\langle \mathrm{ad}^*_X l, \boldsymbol{\nabla} f(l) \right\rangle$$

and therefore the vector field itself can be expressed as

$$\xi_X = -\left\langle \mathrm{ad}^*_X\, l, e_i \right\rangle \frac{\partial}{\partial l_i} = \left\langle e^j, [X, e_i] \right\rangle l_j \frac{\partial}{\partial l_i}. \tag{2.22}$$

These Hamiltonian vector fields are generated by the following linear function

$$f_X(l) \coloneqq \langle l, X \rangle$$

which does indeed yield $\boldsymbol{\nabla} f_X(l) = X$. By the definition of the KK bracket then we have

$$\{f_X, h\}(l) = \xi_X h(l).$$

We can define a closed two form ω on the orbit as

$$\omega_l(\xi_X, \xi_Y) \coloneqq \{f_X, f_Y\}(l) = \langle l, [X, Y] \rangle = f_{[X,Y]}(l).$$

Evaluating this two form at the point m itself we see that $\ker \omega_m \simeq \mathfrak{g}_m$. Which means that our two form is non-degenerate on the orbit. Therefore, the KK bracket defined on any coadjoint orbit is precisely one of the above defined symplectic leaves.

2.2.2. Hamiltonian Reduction

The idea of reduction is a powerful method to construct dynamical systems. Conceptually the idea is to take a system with phase space $\mathscr{P}$ with a Hamiltonian $\mathcal{H}$ which is invariant under the action of a continuous symmetry group. Nöther's theorem yields a corresponding set of integrals of motion. Reduction is the elimination of degrees of freedom by setting these integrals to constant values. Although the dynamics confined to this portion of phase space is often degenerate and so we would also need to eliminate redundant degrees of freedom.

We first turn our attention to proposing a well defined notion of Hamiltonian action of a Lie group. Suppose that our phase space $\mathscr{P}$ is a connected manifold which is endowed with the action of a Lie group G i.e. $G \times \mathscr{P} \to \mathscr{P}$. We denote as gx the image of $x \in \mathscr{P}$ under the action of $g \in G$. This action induces an action on the space $\mathscr{F}(\mathscr{P})$ as

$$T(g)f(x) \coloneqq f(g^{-1}x).$$

In this representation, the vector field corresponding to an element of the Lie algebra is defined in the usual way. So $X \mapsto \xi_X$ defines the Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{X}(\mathscr{P})$. The action is called hamiltonian if, for all $X \in \mathfrak{g}$ there exists a function f_X such that

$$i_{\xi_X}\omega + \mathrm{d}f_X = 0.$$

Thus, the functions f_X generate symplectomorphisms [3]. To further pursue this notion of reduction we need to generalise the results of the previous section slightly.

[3] We note that these functions are defined up to a constant. We choose this constant such that f_X has linear dependence on X and hence $f_{[X,Y]} = \{f_X, f_Y\}$.

Definition 8 (Momentum Map)**.** *For any $x \in \mathscr{P}$ the aforementioned map $X \mapsto f_X$ defines a linear function on $\mathfrak{g}$ i.e. precisely an element $\mu \in \mathfrak{g}^*$,*

$$\langle \mu(x), X \rangle =: f_X(x). \tag{2.23}$$

Thereby defining a Poisson map

$$\mu : \mathscr{P} \to \mathfrak{g}^*. \tag{2.24}$$

This is the famed momentum map named for its analogous properties to momentum and angular momentum as found in classical mechanics[4].

Assuming the linearity condition mentioned previously the momentum map is uniquely defined and the group action defines the vector field

$$\xi_X =: \{\langle \mu, X \rangle, \star\}. \tag{2.25}$$

Since the composition of linear operations, of which the momentum map definition and all Poisson brackets are examples of, is a commutative process

$$\xi_X f = \langle \{\mu, f\}, X \rangle$$

The question now becomes, how do we explicitly define the dual space with a PB. The linearity condition

$$\{f_X, f_Y\} = f_{[X,Y]}$$

gives us precisely the tool we need. Looking at the left hand side and recalling (2.23) we see

$$\{f_X, f_Y\} = \{\langle \mu_1, X \rangle, \langle \mu_2, Y \rangle\}$$

where μ_i denote different copies of the momentum map. Expanding this,

$$\{f_X, f_Y\} = \xi_X f_Y = \langle \{\mu_1, \langle \mu_2, Y \rangle\}, X \rangle = \langle \{\mu \otimes \mu\}(x), X \otimes Y \rangle, \quad \{\mu \otimes \mu\} := \{\mu_1, \mu_2\}. \tag{2.26}$$

Turning now to the right hand side let's express the commutator of two vectors

$$\begin{gathered}[X, Y] = [X, Y]^i e_i = e_i X^j e_j Y^i - (X \leftrightarrow Y), \\ X^i = X e^i \implies [X, Y] = X \otimes Y e^i \wedge e^j [e_i, e_j].\end{gathered}$$

Hence, the right hand side can be written as

$$f_{[X,Y]} = \langle \mu(x), [X, Y] \rangle = \Big\langle \langle \mu(x), [e_i, e_j] \rangle \, e^i \wedge e^j, X \otimes Y \Big\rangle. \tag{2.27}$$

[4]In the literature this object is often, nigh exclusively, called the *moment map*. However, this is apparently an inaccurate translation of the French "application moment". The idea was introduced in full by Jean-Marie Souriau in [9] with roots going back to Lie. Marsden and Weinstein are responsible for this inaccuracy due to their 1974 work [10]. The issue is due to the French use of phrases that would translate to "linear moment" and "angular moment" which simply are not used in English [11]. To attempt to rectify this mistake, no doubt in vain, we shall use only the correct version here.

Therefore, we define the Poisson bracket as

$$\{\mu \otimes \mu\} := \langle \mu(x), [e_i, e_j] \rangle \, e^i \wedge e^j.$$

We are now in a position to notice quite an important point. Define $g(t)$ the one parameter subgroup generated by $X \in \mathfrak{g}$. Then the transformation $x \to g^{-1}x$ is, by definition, generated by the vector field ξ_X. Hence,

$$\langle \xi_X \mu, Y \rangle = \xi_X f_Y = f_{[X,Y]} = \langle \mu, [X, Y] \rangle$$

which tells us that

$$\xi_X \mu = -\operatorname{ad}^*_X \mu. \tag{2.28}$$

Exponentiating this action also yields

$$\mu(gx) = \operatorname{Ad}^*_X \mu. \tag{2.29}$$

This tells us that the momentum map builds a correspondence between the group action on the phase space and the coadjoint action such that an orbit in the phase space is mapped to a coadjoint orbit in the dual space. Now we can define precisely what the reduced phase space actually is.

Definition 9 (Reduced Phase Space)**.** *Define the stabiliser group G_m of a point $m \in \mathfrak{g}^*$ and consider the inverse image $\mu^{-1}(m) \in \mathscr{P}$. Thus, we define the reduced phase space as*

$$\mathscr{P}_r := \mu^{-1}/G_m. \tag{2.30}$$

Theorem 2 (Marsden-Weinstein)**.** *If a point $m \in \mathfrak{g}^*$ is chosen such that the action of G_m is free and proper. That is to say, there exists no fixed points and the topological structure of the space is preserved respectively. Then the reduced phase space is a smooth manifold.*

The Marsden-Weinstein theorem is a well known result and the proof can be found in [10]. The same body of work further asserts claim 1.

Claim 1. *The reduced phase space is a symplectic manifold with symplectic structure induced by ω on the entire phase space.*

Proof. In general, we can evaluate the two form ω_m with the vector fields $V \in T_m\mathcal{O}_m$ and $W \in \mu^{-1}(m)$. Any other evaluation can take this result as a basis. We say that V is generated by some f_V but since f_V is constant on $\mu^{-1}(m)$ then we have

$$\omega_m(V, W) = -\,\mathrm{d}f_V\,(W) = 0.$$

That is to say, along W f_V is constant and hence its derivative is null. Furthermore, since ω_m is invariant under the action of G_m i.e. our choice of representative doesn't matter. The form is well-defined, bilinear, and antisymmetric.

Showing that the form is closed is trivial by its restriction. But we still need to show that the restricted form is well-defined. By the above argument the vertical vectors are precisely those orthogonal to $T_m\mu^{-1}(m)$ i.e.

$$(T_m G \cdot m)^\perp = T_m\mu^{-1}(m).$$

We conclude that this is an equality since both sides have the same dimension of $\dim \mathscr{P} - \dim G$. The left hand side since a G orbit has dimension $\dim G$ so then the skew orthogonal compliment has dimension $\dim \mathscr{P} - \dim G$. The right hand side since $\mu^{-1}(m)$ is defined by $\dim G$ equations. So then we can write the restricted kernel as

$$\ker \omega|_{\mu^{-1}(m)}(m) = T_m\mu^{-1}(m) \cap (\mu^{-1}(m))^{\perp} = \mathfrak{g}_m(m).$$

All of these vectors will project to the identity when we take the quotient with respect to G_m. Therefore $\omega|_{\mathscr{P}_r}$ is non-degenerate. $\square$

We have successfully been able to reduce the dimensionality of the phase space to

$$\dim \mathscr{P}_r = \dim \mathscr{P} - \dim G - \dim G_m.$$

But we promised that we would be able to connect this to the concepts of Nöther's theorem and hence be able to use it in a physical sense. Indeed, if a given dynamical system's Hamiltonian is invariant under the action of a symmetry group G then for any vector $X \in \mathfrak{g}$ we have

$$-\xi_X \mathcal{H} = \{\mathcal{H}, f_X\} = \frac{\mathrm{d}f_X}{\mathrm{d}t} = 0. \tag{2.31}$$

Et voilà, we have shown that for some time parameter t the functions f_X are in fact integrals of motion. Therefore, the momentum map is constant along physical trajectories - earning its name. To obtain the emergent dynamics we have constructed we must venture a bit out into the world of constrained Hamiltonian system. Since if this is not a *constrained* - by the conservation of the momentum map - *Hamiltonian* - by virtue of the hamiltonian action - *system*, what could be? This is a topic which, unfortunately, often goes underappreciated in the context of classical mechanics and is often only introduced in the context of quantum electrodynamics. For completeness a brief explanation can be found in Appendix B since an understanding of Dirac brackets, et cetera has been assumed. An excellent resource to be used for a more in depth discussion is [12]. Of course, we must also mention that Dirac's lectures on quantum mechanics [13] are renowned on the topic for very good reason. However, in keeping with Dirac's personality they are very condensed. We now understand the use in principle and in practice of the Marsden-Weinstein Theorem.

2.2.3. Example: The Cotangent Bundle of a Lie Group

The cotangent bundle of a given simply connected Lie group provides a classic example of a phase space with non-Abelian symmetries. It is what we will consider for our first example of Hamiltonian reduction. This explanation shall not be as thorough as can be. We will only use it to outline the use of reduction and some notation for later on. Suppose our group is, G, has some Lie algebra $\mathfrak{g}$. Define two diffeomorphisms,

$$L_g(h) := gh, \qquad R_g := hg$$

the first called the left action and the second the right action. The differential, or pushforward, of the left action then send $T_hG \to T_{gh}G$. In which case setting $h = e$, the identity element, yields the map $\mathfrak{g} \to T_gG$. Which induces the isomorphism,

$$T_*G \simeq G \times \mathfrak{g}. \tag{2.32}$$

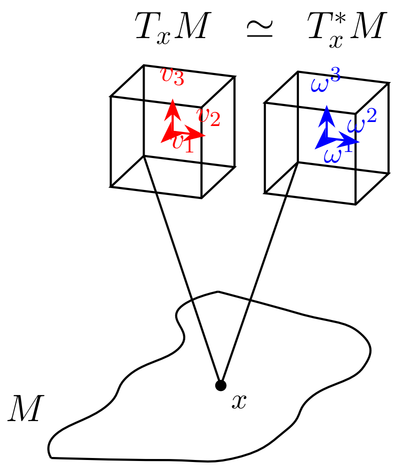

Figure 2.1. Here we see a diagrammatic representation of the tangent space and cotangent space of a given certain point x on a manifold M. The tangent space, on the left and in red, has orthogonal basis elements v_i. The cotangent space, on the right and in blue, has basis elements ω^i. A covector is a linear functional which maps a vector in tangent space to a scalar. Here we specifically mention that the two spaces are isomorphic. This identification can often be accomplished with the definition of an inner product as is easily seen through Dirac notation. It is precisely in this fashion which we will identify the Lie algebra of a Lie group with it's dual space towards the end of Sec. 2.2.3. The tangent, and hence cotangent, bundle of a manifold denoted as TM and T^*M respectively can then just be thought of as the union of all of these individual spaces.

Then we define the left and right invariant vector fields,

$$\xi^l_X := (L_g)_* X, \qquad \xi^r_X := (R_g)_* X, \tag{2.33}$$

for $X \in \mathfrak{g}$. We will largely only cover the left action here and refer to [2] for details on the right action analysis. That being said, we will cite results for the right action when needed.

The Lie group itself is defined with the multiplication and inverse rules,

$$\psi_{g,l}(h,m) := (gh, \operatorname{Ad}^*_{h^{-1}} l + m), \quad \varphi(g,l) := (g^{-1}, -\operatorname{Ad}^*_g l).$$

Then we shall calculate the left vector field as,

$$\frac{\mathrm{d}}{\mathrm{d}t} f(e^{tX} \cdot (g,l))\Big|_{t=0},$$

by using the multiplication map,

$$\begin{aligned} \xi^l_X f(g,l) \; \frac{\mathrm{d}}{\mathrm{d}t} f\Big((e^{tX} g, l)\Big)\Big|_{t=0} &\approx \frac{\mathrm{d}}{\mathrm{d}t} f\Big(((1-tX)\cdot g, l)\Big)\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} f\Big((g(1-tg^{-1}Xg), l)\Big)\Big|_{t=0} \\ &\approx \frac{\mathrm{d}}{\mathrm{d}t} f\Big((e^{-t\operatorname{Ad}^*_g X}, l)\Big)\Big|_{t=0} = \mathscr{D}^r_{-\operatorname{Ad}^*_g X} f, \end{aligned}$$

and so we arrive at

$$\begin{aligned} \xi^l_X f(g,l) &= -\Big\langle e^i, g^{-1}Xg \Big\rangle \mathscr{D}^r_i f \\ \xi^r_X f(g,l) &= \Big\langle e^i, X \Big\rangle \mathscr{D}^r_i f - \Big\langle \operatorname{ad}^*_g l, e_i \Big\rangle \partial^i f \end{aligned}. \tag{2.34}$$

Where the latter is the general formula for the right invariant vector field.

The left invariant field also then defines the right differential of G,

$$\xi^l_X f(g) = \frac{\mathrm{d}}{\mathrm{d}t} f(ge^{tX})\Big|_{t=0} =: \mathscr{D}^r_X f. \tag{2.35}$$

This gives us the commutators in the expected form,

$$\begin{aligned}
[\xi^l_X, \xi^l_Y] = \xi^l_{[X,Y]}, \quad [\xi^r_X, \xi^r_Y] = -\,\xi^r_{[X,Y]}, \quad [\xi^l_X, \xi^r_Y] = 0. \\
[\mathscr{D}^l_X, \mathscr{D}^l_Y] = -\mathscr{D}^l_{[X,Y]}, \quad [\mathscr{D}^r_X, \mathscr{D}^r_Y] = \mathscr{D}^r_{[X,Y]}, \quad [\mathscr{D}^l_X, \mathscr{D}^r_Y] = 0.
\end{aligned} \tag{2.36}$$

The pullbacks also then define the cotangent space and thus the isomorphism[5],

$$T^*G \simeq G \times \mathfrak{g}^*. \tag{2.37}$$

Consider the two differential one forms on the cotangent bundle defined as the left and right invariant forms θ and ϵ such that

$$(L_g)^*\theta(X) =: l(X), \qquad (R_g)^*\epsilon(X) =: m(X), \qquad X, l, m \in \mathfrak{g}^*.$$

As per usual, we assume that G is a matrix Lie group and introduce the *Maurer-Cartan* one forms, Ω^l and Ω^r, which takes values in the Lie algebras. These forms send $T_gG \to \mathfrak{g}$,

$$\Omega^l_g := g^{-1}\,\mathrm{d}g\,(\xi^l_X) = X, \qquad \Omega^r_g := \mathrm{d}g\; g^{-1}(\xi^r_X) = X, \qquad X \in \mathfrak{g}. \tag{2.38}$$

We define the canonical pairing between the Lie algebra and its dual by $\langle \cdot, \cdot \rangle$ and gain formulae for the invariant one forms by,

$$\theta_g = \left\langle l, g^{-1}\,\mathrm{d}g \right\rangle, \qquad \epsilon_g = \left\langle m, \mathrm{d}g^{-1}\; g \right\rangle. \tag{2.39}$$

This shows explicitly the fact that $\theta, \epsilon \in G \times \mathfrak{g}^* \simeq T^*G$. Though θ lives in the left parameterisation of the cotangent bundle and ϵ in the right. These parametrisations can be mapped to each other by,

$$m = \mathrm{Ad}^*_g\, l,$$

then by recalling (2.19) we identify

$$\epsilon_g(X) = \left\langle l(g^{-1}Xg), \mathrm{d}g\; g^{-1} \right\rangle = \left\langle l(X), g^{-1}\,\mathrm{d}g\; gg^{-1} \right\rangle = \left\langle l(X), g^{-1}\,\mathrm{d}g \right\rangle = \theta_g(X),$$

which tells us that $\epsilon(g, m) \sim \theta(g, l)$. Following this duality of left and right parametrisations, we can define the symplectic manifold using either one forms. We will use the left and define

$$\omega := \mathrm{d}\theta \implies \omega_g = \left\langle \mathrm{d}l \wedge g^{-1}\,\mathrm{d}g \right\rangle - \left\langle l, g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g \right\rangle. \tag{2.40}$$

The reader might see $g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g$ and be slightly concerned that this is not equivalently zero by anti-symmetry. This is not true. As we discussed, G is a matrix group and hence the wedge product here is between entries of the matrices. Now defining the function $f(g, l) \in \mathscr{F}(G \times \mathfrak{g}^*)$, the differential can be written, using the left differential, as

$$\mathrm{d}f = \left\langle e^i, g^{-1}\,\mathrm{d}g \right\rangle \mathscr{D}^r_i f + \mathrm{d}l_i\, \partial^i f, \qquad \mathscr{D}^r_i := \mathscr{D}_{e_i}.$$

The left differential here, since it's linear, allows the transfer of the structure constants c^k_{ij} of $\mathfrak{g}$ in a given basis $\{e_i\}$. Looking for the hamiltonian vector field corresponding to this function we suppose the form through an ansatz,

$$\xi_f = \Psi^i \mathscr{D}^r_i + \chi_i \partial^i, \tag{2.41}$$

[5]We here mean in a geometric sense. As will be explained later, this product is actually a semi-direct product. This yields information regarding the group structure only and therefore does not affect this statement.

which tells us,

$$i_{\xi_f}\omega_g = \left\langle \mathrm{d}l\,(\xi_f) \wedge g^{-1}\,\mathrm{d}g \right\rangle - \left\langle \mathrm{d}l \wedge g^{-1}\,\mathrm{d}g\,(\xi_f) \right\rangle - \left\langle l, g^{-1}\,\mathrm{d}g\,(\xi_f) \wedge g^{-1}\,\mathrm{d}g \right\rangle + \left\langle l, g^{-1}\,\mathrm{d}g \wedge g^{-1}\,\mathrm{d}g\,(\xi_f) \right\rangle,$$
$$i_{\xi_f}\omega_g = \left\langle \chi_i e^i, g^{-1}\,\mathrm{d}g \right\rangle - \left\langle \mathrm{d}l\,, e_i \Psi^i \right\rangle - \left\langle l, e_i \Psi^i g^{-1}\,\mathrm{d}g \right\rangle + \left\langle l, g^{-1}\,\mathrm{d}g\, e_i \Psi^i \right\rangle,$$
$$i_{\xi_f}\omega_g = \left\langle \chi_i e^i - \Psi^i\,\mathrm{ad}^*_{e_i} l, g^{-1}\,\mathrm{d}g \right\rangle - \mathrm{d}l_i\,\Psi^i.$$

Where we've used the fact that Ψ^i is an element of the algebra and χ an element of the dual space. By the definition of a hamiltonian vector field, we have $i_{\xi_f}\omega = -\,\mathrm{d}f$ and so we can express the same object as

$$i_{\xi_f}\omega_g = -\,\mathrm{d}f = -\left\langle e^i, g^{-1}\,\mathrm{d}g \right\rangle \mathscr{D}^r_i f - \mathrm{d}l_i\,\partial^i f.$$

Matching these terms we get,

$$\Psi^i = \partial^i f, \qquad \chi_i e^i = \partial^i f\,\mathrm{ad}^*_{e_i} l - e^i \mathscr{D}^r_i f \implies \chi_i = \langle l, [\nabla f, e_i] \rangle - \mathscr{D}^r_i f. \tag{2.42}$$

Thereby the hamiltonian vector field itself is

$$\xi_f = \partial^i f\,\mathscr{D}^r_i + (\langle l, [\nabla f, e_i] \rangle - \mathscr{D}^r_i f)\,\partial^i. \tag{2.43}$$

So we may finally write the Poisson bracket as

$$\{f, h\}(g, l) = \langle l, [\nabla f, \nabla h] \rangle + \langle \mathscr{D}^r h, \nabla f \rangle - \langle \mathscr{D}^r f, \nabla h \rangle\,. \tag{2.44}$$

Suppose now we choose a faithful representation of the Lie group G, ρ, and so we choose to denote the matrix elements of a group element g in this representation by $\rho_{ij}(g)$. So we generalise the computation of Poisson brackets of function on T^*G. Since $\mathscr{F}(G \times \mathfrak{g}) \simeq \mathscr{F}(G) \otimes \mathscr{F}(\mathfrak{g})$ we can build up a basis for the total function space by finding bases for the individual function spaces. We use the elements $\rho_{ij}(g)$ as the basis of the group function space and ξ^i_X for some given $X \in \mathfrak{g}$. The Poisson bracket of any two functions on the phase space is then effectively equivalent to the Poisson brackets of any two combinations of basis functions. Then since $\rho_{ij}(g)$ has no dependence on the momentum - i.e. the l coordinate - then (2.44) simply vanishes.

Next we consider the Poisson bracket of ξ^l_X and ρ. The former, generating translations by some X, acts on a function as a left action $\rho(g) \mapsto \rho(gX)$. Which tells us that this should reduce to a left action. Indeed using (2.44),

$$\{f_X, \rho_{ij}\} = \langle \mathscr{D}^r f_X, \rho_{ij}(g) \rangle = \langle X, \rho_{ij}(g) \rangle = \rho_{ij}(gX).$$

Finally, the Poisson bracket between any two hamiltonian vector field Poisson commute in the usual sense. These results tells us that for some matrices representing group elements ρ and some X corresponding to hamiltonian vector fields ξ_X have the Poisson commutation relations,

$$\begin{aligned} \{\rho, \rho\} &= 0, \\ \{X, \rho\} &= \rho, \\ \{X, X\} &= X. \end{aligned} \tag{2.45}$$

These commutation relations implies that for the phase space $\mathscr{P} = T^*G$ we see that

$$T_e\mathscr{P} \simeq \mathfrak{g} \oplus_s T_e\mathfrak{g}^*,$$

where $\oplus_s$ represents a semi-direct sum. Exponentiating this we have,

$$T^*G \simeq G \ltimes \mathfrak{g}^*,$$

we have the structure of a semi-direct product on our phase space.

Comparing (2.43) to (2.35) we see that we can identify that for the left representation we have that the $\partial^i f$ component is null. The remaining part is given by,

$$\partial^i \left\langle \mu^l, X \right\rangle = -\left\langle e^i, g^{-1}Xg \right\rangle \implies \left\langle \mu^l, X \right\rangle = -\left\langle l, g^{-1}Xg \right\rangle = -\left\langle \mathrm{Ad}_g^* l, X \right\rangle.$$

Applying the same logic to the right invariant vector field we get,

$$\langle \mu^r, X \rangle = \langle l, X \rangle.$$

These leads us precisely to the momentum maps in the left and right representation. Utilising (2.2.3) we get an expression for the right and left moment maps as,

$$\mu^l(g,l) = -\mathrm{Ad}_g^* l = -m, \qquad \mu^r(g,l) = l. \tag{2.46}$$

As we've mentioned before, the reduction technique is oft not complete when constructing the momentum map. Indeed, this is the case here. We see that the left and right momentum maps Poisson commute and thus are both skew-orthogonal with respect to ω at any point of the phase space. This gives two symplectic leaves $\mu_l^{-1}(l)$ and $\mu_r^{-1}(n)$ for any $l, n \in \mathfrak{g}^*$. Fixing some $l = l_0$ we have that since $\mu_l(g,l) = -\mathrm{Ad}_g^* l$ we get

$$\mu_l^{-1}(l_0) = \{(g, -\mathrm{Ad}_{g^{-1}}^* l_0) \mid g \in G\} \simeq G. \tag{2.47}$$

So that the reduced phase space is then given by the left coset with the isotropy group of l_0,

$$\mathscr{P}_r \simeq G_{l_0} \backslash G,$$

and similarly for the right action.

All of this discussion regarding the coadjoint representation and covectors in general begs the question - can we simplify our life by relating this back more directly to the adjoint representation and vectors? Indeed, this is a very natural question to ask. Since introductory linear algebra, we have often made this identification without knowing as much. We would like to discover a bilinear, symmetric, non-degenerate form - also known as an inner product - denoted as $\langle , \rangle_{\mathfrak{g}}$. It should also be invariant under the adjoint action of a group member. Identifying the tangent basis elements with the cotangent basis via an orthonormal inner product relation one can write the Poisson structure outlined in (2.45) explicitly as in [2],

$$\begin{aligned} \{g_1, g_2\} &= 0, \\ \{l_1, g_2\} &= g_2 C_{12}, \\ \{l_1, l_2\} &= \frac{1}{2}\{C_{12}, l_1 - l_2\}. \end{aligned} \tag{2.48}$$

Now l is an element of the Lie algebra and the subscripts refer to matrix spaces as was the case in Sec. 2.1.2. The object C_{12} here is of great significance and will feature again in our discussion on quantum integrability.

Definition 10 (Split Casimir)**.** *Let $\mathfrak{g}$ be a simple Lie algebra with generators X_a and $\mathcal{U}(\mathfrak{g})$ is its universal enveloping algebra. The adjoint representation of the Lie algebra is used to define the Killing form of the algebra, g_{ab}. Using the definition of the quadratic Casimir operator as inspiration together with the fact that a simple Lie algebra is endowed with an invertible Killing form, we define the split Casimir operator,*

$$C = g^{ab} X_a \otimes X_b. \tag{2.49}$$

For further discussion on the split Casimir operator we direct readers to [14].

The action of the split Casimir on an operator that lives in the mixed matrix space, e.g. $B_{12} = \sum_{ij,\ kl} B_{ij,\ kl} E_{ij} \otimes E_{kl}$, is to permute the matrix spaces. That is to say,

$$C_{12}B_{12} = B_{21}C_{12}.$$

In particular, when concerning our group elements g_i in the context of the invariance of the adjoint action we have that the split Casimir to feature for us is of the form,

$$C_{12} = \sum_i e_i \otimes e_i. \tag{2.50}$$

Thus concludes our discussion on symplectic geometry and with it our overview of classical integrability. The mathematical framework we have developed the basis of has allowed us a mere glimpse through the rabbit hole to the Wonderland of integrability. We again must recommend [1], [2] for further detail and discussion. Indeed, our rationale for exploring the cotangent bundle of a Lie group as our limit and most detailed venture into the mathematical realm of classical integrability is purely to be able to utilise Hamiltonian reduction on $T^*GL_N(\mathbb{C})$ to produce the hyperbolic Calogero-Moser-Sutherland (CMS) model and its relativistic[6] generalisation the Ruijsenaars-Schneider (RS) model. The former being of particular interest throughout this text.

[6] As is always necessary to caution, the term *relativistic* here does not naïvely relate to special relativistic physics. Indeed, it is defined for particles with a potential that necessarily travels instantaneously, the so called "spooky action at a distance", and therefore to consider it relativistic in nature is fundamentally incorrect. Instead, relativistic here is merely in the context of generating the Poincaré algebra.

3. Quantum Integrability

3.1. Quantum Groups

3.1.1. Quantisation

As we have found through our lengthy discussion in Sec. 2.2, the mathematics underlying integrability is difficult. It requires a keen understanding of algebra and geometry. This fact is demonstrated even more so in the latter sections on doubles and Poisson reduction. It is ironic that classical mechanics - taught from a school age and the shared basis for humanity's interpretation of reality - can be so intuitive in nature but so incredibly demanding in its requisite mathematics whilst quantum mechanics - famed for its unintuitive results and loved by sensationalist pop sciences authors the world over - can actually be relatively simple in the pursuit of solutions. Indeed, one cited reason for this is the highly non-linearity found in classical equations whereas the basis for non-relativistic, closed quantum mechanics is the *linear* Schrödinger equation. In our following discussion, we will eventually arrive at the stage where our work is easier than the corresponding classical analysis would be. However, before we get there we must delve once more unto the breach.

We begin by first simply generalising the notion of a Lie group to a structure that makes a natural extension.

Definition 11 (Poisson Map)**.** *Let $\mathscr{P}$ and $\mathscr{Q}$ be Poisson manifolds with Poisson bracket $\{\cdot,\cdot\}_P$ and $\{\cdot,\cdot\}_Q$ defined on them respectively. Then a map $\varsigma : \mathscr{P} \to \mathscr{Q}$ is considered to be a Poisson map if for all $x \in \mathscr{P}$ and functions $f, g \in \mathscr{F}(\mathscr{Q})$ the following condition holds,*

$$\{f, g\}_Q(\varsigma(x)) = \{f \circ \varsigma, g \circ \varsigma\}_P(x).$$

Definition 12 (Poisson-Lie group)**.** *A Poisson-Lie group is a Lie group G together with a Poisson bracket where the manifold corresponding to the group is Poisson. Furthermore, the map defining group multiplication, $\psi : G \times G \to G$ must be a Poisson map.*

This notion of a Poisson-Lie group is central to classical and quantum integrability. The former being a natural arena to analyse Poisson reduction - a generalisation of the Hamiltonian reduction discussed in Sec. 2.2.2 - which is beyond the scope of this text but the reader is referred to [2]. The latter is the motivation behind defining it here. We intend to "quantise" the Poisson-Lie group.

A seemingly common practice is the conversion of the real space dynamical variables, position and momentum, with holomorphic counterparts. In which case we also define $\hbar$ not as unity, as God intended [15], but to some complex number $\hbar \in \mathbb{C}$. This decision can be taken for several reasons. When analysing the quantum Hall effect, for example, since the covariant derivative includes an imaginary term being the vector potential then it is natural to work on the complex plane. In our case, the reduction technique is actually blind to a lot of constraints and actually produces what's called an *algebraic integrable system*. Once this has been constructed, one can enforce physical conditions. So here we find that it's simpler to work with complex dynamical variables but one can always enforce the hermiticity of these operators together with $\hbar \to -i\hbar$ some purely imaginary number. In the holomorphic regime, the canonical commutation relations have adjusted as follows,

$$[q_i, q_j] = 0, \qquad [p_i, p_j] = 0, \qquad [p_i, q_j] = \hbar\delta_{ij}. \tag{3.1}$$

Although we have spent a substantial amount of time and effort building up the fundamentals of integrability through classical mechanics, we must still be very careful about what intuition can and cannot be carried through to the quantum regime. Indeed, for a Liouville integrable system with involutive integrals of motions f_k then we say the corresponding quantum integrable system is the quantisation of the classical system and we impose that the operators F_k will all commute. Similarly, following the fact that the set of functions f_k are all independent we make the claim.

Claim 2. *The set of quantum operators F_k are not algebraically independent following from their classical counterparts' functional independence.*

Proof. We will prove this by contradiction. Suppose we have some finite dimensional Hilbert space $\mathfrak{H}$ with dimension d. On this Hilbert space we choose a shared eigenbasis of two commuting operators A and B. We can represent these operators in terms of the eigenbasis projection operators, i.e. $A = \sum_j a_j P_j$ with $P_j = |j\rangle\langle j|$, or as matrices

$$A = \begin{pmatrix} a_1 & \dots & 0 \\ \vdots & \ddots & 0 \\ 0 & 0 & a_d \end{pmatrix}, \tag{3.2}$$

with B being decomposed in the same fashion. Since these matrices are diagonal, through repeated substitutions and multiplications we can represent the projection operator as $P_j = \prod_{i \neq j} \frac{B-b_i}{b_j-b_i}$. But since the operator A depends on the projection operator the claim must be incorrect. $\lightning$

3.1.2. Quantum Yang-Baxter

We now move on to provide a very brief definition of Hopf algebras. As always, the text [2] provides an excellent summary of the topic which is more comprehensive than here whilst the more interested reader is directed to [16]. This definition - more precisely a small addition alongside this definition - will grant us the ability to work with the quantum Yang-Baxter equation and with it, the basis of quantum integrability in practice.

Definition 13 (Hopf algebra)**.** *Let $\mathscr{A}$ be a unital associative algebra. We say that $\mathscr{A}$ is a Hopf algebra over a field $\mathbb{K}$ if we can define on it a coproduct $\Delta : \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$, a counit $\epsilon : \mathscr{A} \to \mathbb{K}$, and an antipode $S : \mathscr{A} \to \mathscr{A}$ satisfying a set of axioms which can be found in [2]. Furthermore, we note that a vector space defined with only Δ and ϵ is called a coalgebra whilst an algebra that is also a coalgebra is called a bialgebra.*

Definition 14 (Quantum group)**.** *A quantum group*[1] *is a Hopf algebra over a field* $\mathbb{C}$ *under some particular deformation which produces an additional relation which classifies the algebra as a quasi-triangular Hopf algebra. This additional relation is equivalent to the statement that there exists an object* $R \in \mathscr{A} \otimes \mathscr{A}$ *which is a solution to the quantum Yang-Baxter equation.*

R here should be considered a quantum analogue of the classical r-matrix as defined in (2.13) and so we will often refer to this as the quantum R-matrix or simply the R-matrix. The quantum Yang-Baxter equation, hither-forth referred to just as the Yang-Baxter equation, is a deceptively simple relation,

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{3.3}$$

But do not let this fool you, it is incredibly rich with physical and mathematical curiosities. We will retroactively motivate it with two similar examples relating to the notion of a spin chain and factorised scattering, the latter essentially being the fundamental assumption predicating the Bethe ansatz - one of the most central techniques in quantum integrability. We should also make note of a separate form that the Yang-Baxter can take using the split Casimir operator as defined in 10.

Claim 3. *If and only if a matrix,* R*, satisfies the Yang-Baxter equations, then the matrix* $\mathscr{R} := CR$ *satisfies the braid equation,*

$$\mathscr{R}_{12}\mathscr{R}_{23}\mathscr{R}_{12} = \mathscr{R}_{23}\mathscr{R}_{12}\mathscr{R}_{23}. \tag{3.4}$$

Therefore, the braid equation is equivalent to the Yang-Baxter equation.

The proof is omitted but can be found in [18].

One of the first and most fundamental interacting quantum systems approached by physicists in the Heisenberg spin chain. A model developed by Werner Heisenberg in order to study the statistical physics of magnetic systems [19]. Here we will concern ourselves with the one-dimensional XXX model with periodic boundary conditions which consists of a chain of spin $\frac{1}{2}$ particles, i.e. electrons, that forms a loop as is seen in Fig. 3.1. Without loss of generality, we can also remove the effect of an external magnetic field by gauge transforming resulting in not periodic but twisted boundary conditions. Each electron is static and therefore is described entirely by its spin state thereby forming a single particle two dimensional projective Hilbert space $\mathfrak{H} \simeq S^2$ with basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. Then if we decide to model an N electron long chain the Hilbert space of the entire system is obviously $\mathscr{H} = \bigotimes_{i=1}^{N} \mathfrak{H} = \mathfrak{H}^{\otimes N}$. The Hamiltonian of the system, noting again the static nature of the electron, is given by

$$H = -J \sum_{i=1}^{N} S_i^{\alpha} S_{i+1}^{\alpha}, \tag{3.5}$$

where S_i^{α} is the α^{th} component of the i^{th} electron's spin.

Suppose we want to switch around two of the electrons. How would one accomplish this? One way of looking at the problem is to question the nature of a one dimensional system itself. Obviously we live in a $3+1$ dimensional world so what is the use in discussing one dimensional systems to begin with?

[1]At least here we will consider the term "quantum group" to refer to the term in the sense put forth by Drinfel'd [17]
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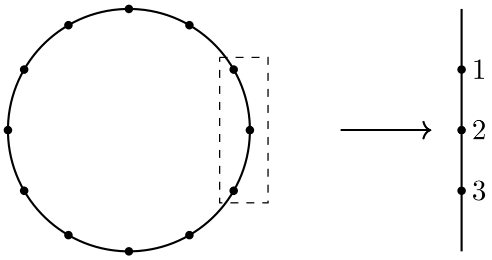

Figure 3.1. An illustrative example of the Heisenberg spin chain with periodic boundaries consisting of $N = 12$ electrons. Highlighted is a segment of 3 electrons which we study more closely in Fig. 3.2 while discussing the R-matrix as it applies to this system.

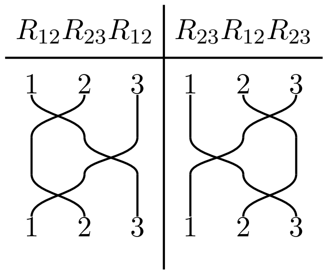

Figure 3.2. A diagrammatic illustration of the concept of braiding as motivated by swapping neighbouring electrons in the XXX Heisenberg spin chain. Here we see that even though the electron pairs are swapped in different orders, it does not change the end result - the action still corresponds to the cycle (1 3). This is the notion of "consistency" that makes the quantum Yang-Baxter equations so useful.

Well, physically the system is still embedded in an ambient higher dimensional space. In our case, we might as well treat this ambient space as two dimensional although again technically it is three dimensional. Through this lens, to swap particles around would require moving the particles in a way that appears to each electron as a relative circular path around it travelled by the other electron. In fact, this perspective is incredibly important in the context of anyons. For example, in the fractional quantum Hall effect when quasi-particles are moved in this fashion their wave functions acquire an additional Berry phase. The result of which is fractional statistics and ultimately fractional charge. However, as mentioned above and seen in (3.5), there are no dynamics in this system and therefore the practicalities of "moving" the electrons is not overly well defined. With a pinch of salt, though, it does physically motivate the second viewpoint of the problem - the introduction of **auxiliary spaces**. An auxiliary space is a purely mathematical object which is a precise copy of the single particle Hilbert space. This is, in fact, the singular space which the R-matrix acts on a tuple tensor producted together. It is these auxiliary spaces that the electrons "live in" whilst being "moved". Identifying R with the swapping of two neighbouring electrons allows us to think of the Yang-Baxter as a consistency equation describing the equivalence of permuting particles in the same fashion in two different ways. In particular, we want to permute three neighbouring electrons via the cycle (1 3). This is illustrated in Fig. 3.2. Returning to the embedding picture briefly, it is this precise relation together with a sense of orientation, i.e. the difference between clockwise and anti-clockwise swaps, that generate the braid group which is in turn so fundamental to the study of anyons.

Before we move on to the second physical use case for the quantum Yang-Baxter, let's briefly introduce the idea of a monodromy matrix. To begin, we connect back to the classical case discussed in Sec. 2.2.3. In particular, since we have shown that $\mathfrak{g} \simeq \mathfrak{g}^*$ then we identify some $l \in \mathfrak{g}$ belonging to an orbit of

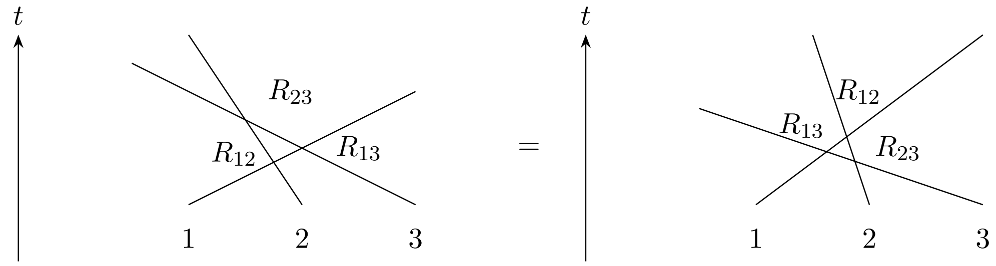

Figure 3.3. Another diagrammatic representation of the quantum Yang-Baxter equation. Specifically, the original form with non-neighbour terms allowed i.e. before utilising the split Casimir operator. The equivalence comes from the indeterminancy in the order of scattering. The left hand shows the term $R_{23}R_{13}R_{12}$ corresponding to a scattering between particles 1 and 2 then 1 and 3 and finally 2 and 3. Whilst the right hand side shows the process of 2 and 3 scattering then 1 and 3 followed by 1 and 2. Hence, it represents the term $R_{12}R_{13}R_{23}$.

$G = GL_N(\mathbf{C})$ with maximal dimension so that it fully describes the system. Then we can diagonalise $l = TQT^{-1}$. This matrix T can be interpreted as relating to the interaction defining a system. In this way we reveal its other name - the *transfer matrix*. As one will recall from quantum statistical physics, the transfer matrix, T, of a system with N particles is used to calculate the canonical partition function, $Z = \text{Tr}\left\{T^N\right\}$.[2] It's clear through this transfer matrix representation that the mondoromy matrix represents a pairwise interaction, in this context between two neighbouring particles. Which allows us to motivate another relevant equation. If two neighbours interact and then are swapped, this is equivalent to first swapping the particles and then allowing them to interact but in the opposite order.

$$R_{12}T_1T_2 = T_2T_1R_{12}. \tag{3.6}$$

This equation is well understood in the context of the spin chain but we will come back to it time and time again.

Now, the second physical motivation. This ties into the work started by Hans Bethe [20] consequently named the *Bethe Ansatz*. The basis for this is the concept of factorised scattering. The ansatz, in this case, just that particles with interact in a purely pairwise fashion. This will be explained in more detail in a later section but for the time being we can note that another assumption we use to simplify the problem. Since the Hamiltonian interactions can be extremely complicated the process for actually computing wave functions and such we deal purely in asymptotic limits. That is to say, supposing that N particles enter in to the system in a certain order and with particular momenta. They will interact in some highly non-trivial fashion and then return to infinity with a new set of momenta. These sets are called the *asymptotic momenta* and are a discrete set of values that are permuted. However, given that we do not actually require to know precisely what happens in the interaction section, particularly what order these pairwise scatterings occur in. For consistency purposes, any ambiguities here must similarly be solved by equation the two cases. This equality is demonstrated in Fig. 3.3.

We have now finally retroactively motivated the existence of quantum groups and, by proxy, given a glimpse of the utility held by the quantum Yang-Baxter equation. Before we move on to a concrete

[2] It's worth noting here the similarity between the partition function's form and the integrals of motion in the Lax representation as outlined in A.3. The left hand side of both cases, Z and I_k respectively, refer to fundamental information about the system. Whilst the right hand has the same trace of a matrix power form. This duality provides us motivation to how we define our objects.

example of a quantum integrable system, let's return try to quantise our earlier idea discussed in 2.1.2, namely that of the spectral parameter. Effectively, by creating the space $W = \bigoplus_{k\in\mathbb{Z}} \lambda^k V$, where λ again plays the role of the spectral parameter, we can define the matrix $R \in \mathrm{Mat}(W \otimes W, \mathbb{C})$ as our new R-matrix hence satisfying the quantum Yang-Baxter relations. The matrix valued function which acts only on two of these spaces in particular, $R(\lambda, \mu) \in V \otimes V$, is represented in the quantum Yang-Baxter equations as

$$R_{12}(\lambda,\mu)R_{13}(\lambda,\tau)R_{23}(\mu,\tau) = R_{23}(\mu,\tau)R_{13}(\lambda,\tau)R_{12}(\lambda,\mu). \tag{3.7}$$

Given a solution to this equation, one can also find an object $T(\lambda)$ in a similar spirit. Performing a Laurent series decomposition on this object yields generators of what is called the quantum affine group. Of course, ensuring the following condition is met so as to not over count,

$$R_{12}(\lambda,\mu)T_1(\lambda)T_2(\mu) = T_2(\mu)T_1(\lambda)R_{12}(\lambda,\mu). \tag{3.8}$$

This concept is used in physical system, namely when considering functions of momenta. However, the formality of infinite dimensional quantum groups is beyond the scope of this text and we refer the interested reader to [16].

3.2. The Calogero-Moser-Sutherland Model

The Calogero-Moser-Sutherland (CMS) model is a bit of a misnomer, since it is in fact a family of integrable models. The rational CMS model is perhaps the simplest and most widely known of the family. It describes the scenario of particles confined to a one dimensional line with an inverse square pairwise potential. Furthermore, we will often consider that all confined in a harmonic potential,

$$H = -\frac{\hbar^2}{2}\sum_i \partial_i^2 + \frac{\omega^2}{2}\sum_i q_i^2 + \frac{\gamma(\gamma-\hbar)}{2}\sum_{i\neq j}\frac{1}{q_{ij}^2}, \tag{3.9}$$

where $q_{ij} := q_i - q_j$ and we often set an effective coupling constant[3] $g := \gamma(\gamma - \hbar)$. In this section we aim to outline three different CMS models and provide a brief discussion of the spectral problem.

First and foremost, the integrability of the CMS models is shown stemming from a most interesting but perhaps unintuitive of places. We have discussed the procedure of Hamiltonian reduction on the cotangent bundle of a Lie group in some detail in 2.2.3. One finds that from solving for the momentum map and that there is at least two families of involutive functions that can be used to reduce the phase space. The first of which results, through the reduction procedure, in the rational RS model and the second actually results in the *hyperbolic* CMS model. Acquiring the Lax pair of this system we can search for integrals of motion. Doing so, we realise that this member of the CMS family is simple generalisation of the rational case. In particular, one can represent the Hamiltonian of system in a similar fashion as (3.9),

$$H = -\frac{\hbar^2}{2}\sum_i \partial_i^2 + \frac{g}{2}\sum_{i\neq j}\frac{1}{4l^2\sinh^2(q_{ij}/2l)},. \tag{3.10}$$

[3] One may notice that this coupling constant is quadratic in the parameter γ and ask what might be different when the system decouples the inverse square ppotential in these ways. This question is related to statistics and the answer will be discussed later on.

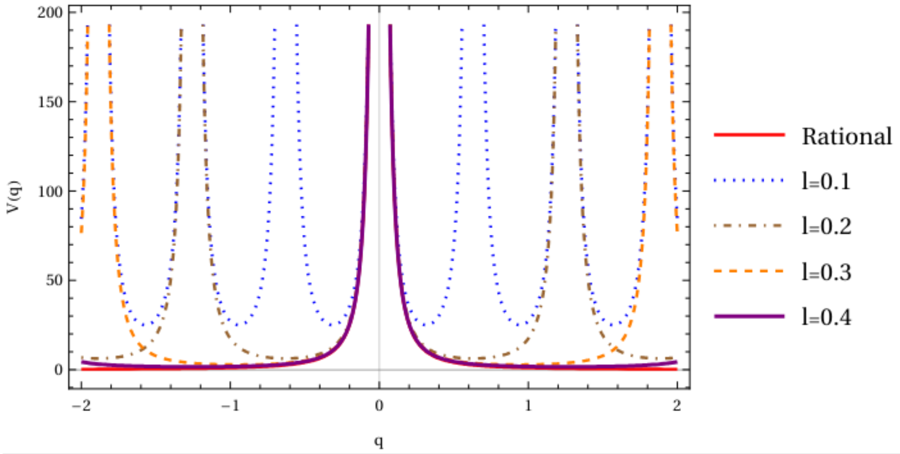

Figure 3.4. A comparison between the pure rational CMS potential with no confining harmonic term with the trigonometric CMS potential for varying characteristic lengths. Here we have set the coupling $g = 2$ and we are dealing with the two particle case. Furthermore, we work in coordinates of the first particle q with some interaction potential $V(q)$ in each case where we have set our coordinates such that the second particle is fixed at $q = 0$.

The mathematics involved in this procedure are lengthy, complicated, and ultimately irrelevant to our discussion, therefore we refer the reader to [2] for more details.

As we can see, expanding in large characteristic lengths this simply reduces back to the rational case. For our discussion, we will primarily focus on the rational model, but before discussing it any further, let's showcase our two other CMS models. Although we won't discuss their properties nor spectra in detail, they're existence is vital to one of the applications of the CMS model and we will briefly discuss - namely the quantum Hall effect on varying geometries.

The trigonometric CMS model is often viewed as a finite size version of the rational CMS model. It can be obtained by analytically continuing the real space position coordinates $q_i \mapsto iq_i$. The Hamiltonian, then, becomes

$$H = -\frac{\hbar^2}{2}\sum_i \partial_i^2 + \frac{g}{2}\sum_{i\neq j}\frac{1}{4l^2\sin^2\frac{q_{ij}}{2l}}. \tag{3.11}$$

Notice here that even to find bound states, we do not need the confining harmonic potential that we had in (3.9). The characteristic length in this case, l once more, is of great physical importance. It effectively describes the finite size localisation that the potential enforces. Similarly, to the hyperbolic case - though a bit more physically intuitive - expansions in large confining lengths will approach the rational case. This transition can be seen directly in Fig. 3.4.

The final and by far the most complicated of the family is the elliptical CMS model. Its Hamiltonian is given by,

$$H = -\frac{\hbar^2}{2}\sum_i \partial_i^2 + \frac{g}{2}\sum_{i\neq j}\wp(q_{ij}\,|\,\omega_1,\omega_2). \tag{3.12}$$

It can be shown that the rational, hyperbolic, and trigonometric models are all just special cases of the elliptical model with varying values of $\omega_i \in \mathbb{C}$. It is the immense difficulty of this model that inspires a lot of work on the topic and so anything approaching a complete discussion on the topic is outside the bounds of this document. However, we will return to it in a later discussion.

3.2.1. The Spectrum of the CMS Models

Much of quantum mechanics in the real world revolves around the *spectral problem.* That is to say, given a quantum system, how can one find the eigenvalues of the Hamiltonian - the allowed energy levels of the system. It is this central problem which has pushed forward the foundations of quantum mechanics like no other. The canonical formulation in the problem, as taught by physics classes and books the world over, is to find the Hamiltonian of the system in some basis - oft related to position or momentum - and exploit the orthogonality of polynomials to find a spectrum. This is precisely the approach that we will consider here.

We begin first with the rational CMS model in a confining potential, as the simplest example. In the one particle case this simplifies even further down to everyone's favourite integrable system: the harmonic oscillator. The standard procedure for the construction of a polynomial basis which the wave function lives in is used here. Defining first the creation and annihilation operators,

$$a^\dagger := \frac{1}{\sqrt{2}}\left(-\hbar\partial + \omega q\right), \qquad a := \frac{1}{\sqrt{2}}\left(\hbar\partial + \omega q\right), \tag{3.13}$$

allows us to write the Hamiltonian in the beautifully factorised form,

$$H = a^\dagger a + \underbrace{\frac{1}{2}\hbar\omega}_{=:E_0}. \tag{3.14}$$

Solving the differential equation posed by acknowledging that the annihilation operator will kill the ground state i.e. $a\,|\Delta\rangle = 0$ in the position basis results in the well known Gaussian wave packet solution $\Delta(q) \sim e^{-\omega q^2/2\hbar}$. We now construct the excited states in a generalisable fashion. One can define the excitation Hamiltonian operator, $\mathscr{H} := \Delta^{-1}(H - E_0)\Delta$ and show trivially that it has the position space representation,

$$\mathscr{H} = -\frac{\hbar^2}{2}\partial^2 + \hbar\omega q\partial. \tag{3.15}$$

Now, to find eigenfunctions of $\mathscr{H}$ is equivalent to finding the polynomial part of the entire eigenfunctions of H. Provided they take the form of $\Psi_n = P_n\Delta$, with P_n some polynomial. One may notice that if we expand a polynomial over a monomial basis i.e. q^i for $i = 0, \ldots, n$ that $\mathscr{H}$ will act as an upper triangular matrix on this basis. The latter term will not change the order of the monomial whilst the former reduces it two-fold. Expanding over this basis and enforcing that the eigenvalue of the operator equal $\hbar\omega n$ we can get a recursion relation which in turn, when solved, reduces back to the Hermite polynomials.

As mentioned, the utility in this approach is its ability to be generalised to the many particle case. We can generalise the form of a and $a^\dagger$ to a pair of non-Hermitian operators[4],

$$\mathcal{A}_i := \hbar\partial_i + \omega_i - \gamma\sum_{j\neq i}\frac{1}{q_{ij}}, \qquad \mathcal{A}_i^\dagger := -\hbar\partial_i + \omega_i - \gamma\sum_{j\neq i}\frac{1}{q_{ij}}. \tag{3.16}$$

[4] It is crucial to point out that these are *not* creation and annihilation operators since they do not form a canonical commutation relation algebra.

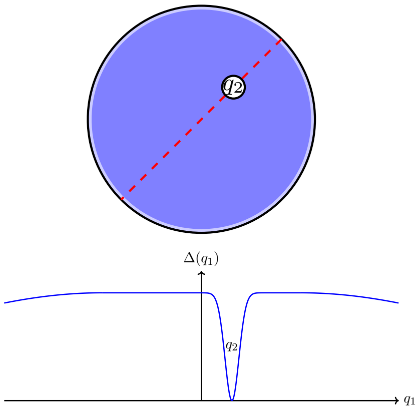

Figure 3.5. A qualitative image showcasing the behaviour of the ground state wave function of the rational CMS model for a two body system. We have a two dimensional geometry representing q_1q_2 space. After fixing a certain value of q_2, the first particle has a particular ground state wave function represented by the dashed red line in the disk and represented by the graph beneath. This representation of the system space will also be referred to again in the quantum Hall effect context.

Then performing precisely the same calculation we can find the ground state wave function again. Defining $\beta := \gamma/\hbar$, we find that the ground state in the position basis is given by,

$$\Delta = \prod_{i<j}(q_i - q_j)^\beta e^{-\sum_i \frac{\omega q_i^2}{2\hbar}}. \tag{3.17}$$

This is an amazing result and should be setting off alarm bells for many the reader with a background in condensed matter theory. This is naught but the Laughlin wave function as seen in the fractional quantum Hall effect! Specifically, for filling factor $\beta = m = \frac{1}{\nu}$. Thus, we can see that from the (anti)symmetry properties of the function - at least at the ground state - if β is odd then we can describe fermions whilst β even will describe bosonic systems. In particular, the case $\gamma = 0$ describes free bosons and $\gamma = \hbar$ describes free fermions. Notice, however, that in either case of statistics the wave function vanishes for $q_i = q_j$ irregardless of statistics. This is a statement of the divergence in the potential term. This eliminates the chances for tunnelling thereby allowing us to fix the order of particles. This is qualitatively demonstrated for $N = 2$ in Fig. 3.5. This is a clue to some very deep connection between the CMS models and the quantum Hall effect that we will discuss later.

Now that we have argued that the particles are, without loss of generality, ordered $q_i \geq q_{i+1}$ forever we can solely focus on this region of solution space, generalising it via particle statistics once a solution has been found. Indeed, from our discussion regarding the generalised Hermite polynomials in C.2.1 we can find the excited state wave function. Following this, we can attain the energy of a state corresponding to a given Young diagram,

$$E_\lambda = \hbar\omega \sum_i \lambda_i + \frac{1}{2}N\hbar\omega + \frac{N(N-1)}{2}\gamma\omega. \tag{3.18}$$

A more physically beautiful way of presenting this result comes from defining the following "momenta",

$$k_j := \lambda_{N+1-j} + (j-1)\beta, \quad \text{satisfying,} \quad k_{j+1} - k_j \geq \beta.$$

This latter inequality can be interpreted as a generalised exclusion principle. Where particles must have momenta separated β i.e. zero for free bosons and unity for free fermions. In which case we can write the energy of the system as,

$$E = \frac{1}{2}\hbar\omega N + \hbar\omega \sum_i k_i. \tag{3.19}$$

This decomposition is incredibly motivating for the study of integrable systems. As is well known, additivity in the energy spectrum corresponds to the separability of the Hilbert space. Therefore, an integrable system is one which can be written in terms of non-interacting quasi-particles carrying quasi-momentum.

Clearly any pair of partitions with the same sum of λ_i or k_i will have the same energy. This degeneracy grows as a factorial which, as one could imagine, leads to highly degenerate states. This degeneracy is actually related to a non-abelian symmetry in the model, specifically that of the W_N-algebra [21]. In fact, degeneracies are the smoke corresponding to the flame of a non-abelian symmetry - think of the hydrogen atom and $SO(4)$ for instance.

Precisely the same procedure can be performed for the trigonometric CMS model [2] utilising now Jack polynomials, which are discussed in [2]. Recall the form of the Hamiltonian given in (3.11) in which one can define the variables, $Q_i := e^{iq_i/l}$. The Hamiltonian then becomes,

$$H = -\frac{\hbar^2}{2}\sum_i \partial_i - \frac{g}{2l^2}\sum_{i\neq j}\frac{Q_iQ_j}{Q_{ij}^2}. \tag{3.20}$$

The ground state wave functions is now given by, up to a phase,

$$\Delta = \prod_{i<j}(Q_i - Q_j)^\beta \prod_i Q_i^{-\frac{1}{2}(N-1)\beta}, \tag{3.21}$$

which should again remind the reader of the Laughlin type wave function. Taking our previous advice on board regarding the utility of quasi-momenta we define,

$$p_j := \frac{\hbar}{l}\left(\lambda_{N+1-j} + (j - \frac{N+1}{2})\beta\right), \tag{3.22}$$

satisfying the exclusion principle, $p_{j+1} - p_j \geq \gamma/l$. With this we can write the energy and total momentum as,

$$E = \frac{1}{2}\sum_i p_i^2, \qquad P = \sum_i p_i. \tag{3.23}$$

In principle, having solved the trigonometric CMS model we can similarly solve the hyperbolic CMS spectrum.[5] So we will not comment on this here. The elliptical CMS model, however, is not so simple. It has well been considered an outstandingly difficult problem. Recently, an explicit solution of the eigenfunctions and spectrum were obtained in the form of infinite series, utilising so called elliptic deformations of the aforementioned Jack polynomials [22].

Before we move on to discuss our final technique in quantum integrability, we must first finally explain these drip-fed connection to the quantum Hall effect. We have seen first hand through the rational CMS model that the state can be essentially boiled down to a one dimensional projection, or slice, through the quantum Hall system on a disk as seen in Fig. 3.5. This notion can be formalised, however,

[5] It's worth pointing out that due to convergence issues, it is not be a trivial leap.

as in [3]. Essentially, this work illustrates that when the quantum Hall effect is put on a disk and one treats the magnetic field as a deformation parameter, that for sufficiently high magnetic fields one can justify projection the entire system down to the lowest energy level - called the Lowest Landau Level (LLL). Physically, this corresponds to the inability of electrons to jump to the next higher band since the band gap is proportional to the magnetic field. This projection effectively halves the dimensionality of a two dimensional system, with a 4 dimensional phase space. The resultant 1 dimensional system can be shown to have the same solution as the rational CMS model. Furthermore, placing the quantum Hall effect on a cylinder one retrieves the trigonometric CMS solutions and even more astoundingly performing a Fourier transform on that solution which amounts to swapping the canonical position and momentum terms results in the solution to the hyperbolic CMS model. The elliptical case is a bit more subtle but this will be discussed later.

3.3. Bethe Ansatz

To understand the use cases of the Bethe ansatz and how it allows us to solve physical models, we first need to build a solid foundation in quantum mechanical time independent factorised scattering theory.

3.3.1. Factorised Scattering

As mentioned previously, much of quantum integrability can be physically broken down, i.e. factorised, into pairwise interactions between particulars. Therefore we must first briefly review two-body scattering.[6]

The base idea of the procedure is to split the wave function into a relative motion part, with momentum k, and centre of mass part, with total momentum K. Justified since the potential is translationally invariant. In this case,

$$\Psi(q_1, q_2) = e^{iK(q_1+q_2)}\psi(x), \quad x := q_1 - q_2. \tag{3.24}$$

So for sufficiently large x we have that the potential disappears and we can form scattering solutions and by sewing them together over different regions form the following sets of asymptotic solutions,

$$\psi_1(k,x) = \begin{cases} e^{-ikx/2} + A(k)e^{ikx/2}, & x \to \infty, \\ D(k)e^{ikx/2}, & x \to -\infty \end{cases}, \quad \psi_2(k,x) = \begin{cases} B(k)e^{-ikx/2}, & x \to \infty, \\ e^{ikx/2} + C(k)e^{-ikx/2} & x \to -\infty. \end{cases} \tag{3.25}$$

The precise formulation of the argument allowing us to cite this general result is based on simple quantum mechanics and can be found in [2], [23]. These texts also detail the procedure, utilising the Wronskian, to match coefficients at the barrier between regions. The nature of working with general potentials can be challenging, we therefore choose to use symmetries to their fullest potential to simplify the problem as much as possible. For instance, by noting that the Schrödinger equation

[6] We forego a discussion on classical scattering here. One should consult [2] for any required further detail but we shall attempt to be as self-contained as possible.

is invariant under the $\mathbb{Z}_2$ symmetry of time reversal we can identify $B = D$. Furthermore, we assume that the particles are sufficiently similar such that the potential is even and hence the Schrödinger equation is also invariant under parity. Which, combined, allow us to introduce the S-matrix,

$$S := \begin{pmatrix} A & B \\ B & A \end{pmatrix}. \tag{3.26}$$

This is the quintessential object in non-relativistic quantum scattering theory. Indeed, one may have a glimpse into its utility as discussed in Sec. 3.1.2. However, the full weight of its use has yet to be explored. We should note that the above assumptions also makes the S-matrix unitary. A basic point that reinforces our interpretation of it as an evolution operator.

As is often the case in integrability, we have taken a complicated problem then rewritten it in terms of non-interacting systems and finally we now recombine the system to find the full solution in terms of the single particle solutions in the asymptotic regions,

$$\Psi(q_1, q_2) \to \begin{cases} c_1 e^{ip_1q_1+ip_2q_2} + (c_1A + c_2B)e^{ip_2q_1+ip_1q_2}, & q_1 \ll q_2, \\ c_2 e^{ip_2q_1+ip_1q_2} + (c_2A + c_1B)e^{ip_1q_1+ip_2q_2}, & q_1 \ll q_2. \end{cases} \tag{3.27a}$$

$$\text{Alternatively,} \quad \Psi(q_1, q_2) \to \begin{cases} \mathcal{A}(12|12)e^{ip_1q_1+ip_2q_2} + \mathcal{A}(12|21)e^{ip_2q_1+ip_1q_2}, & q_1 \ll q_2, \\ \mathcal{A}(21|12)e^{ip_2q_1+ip_1q_2} + \mathcal{A}(21|21)e^{ip_1q_1+ip_2q_2}, & q_1 \ll q_2. \end{cases} \tag{3.27b}$$

These $\mathcal{A}$ objects are the amplitudes of different solution states where in general $\mathcal{A}(\sigma|\tau)$ is the amplitude ascribed to a state in which the positions are given the action of $\sigma \in \mathfrak{S}_2$ on the set $\{1, 2\}$ and the momenta are given by the action of τ in a similar fashion. Indeed, this parametrisation makes physical sense and we can identify the coefficients in (3.27a) and (3.27b) with one another. Doing so tells us that, in fact, of the four amplitudes only two are independent at any time - which brings with the freedom of choosing which two. In a similar fashion to how although Heisenberg and Schrödinger pictures are entirely equivalent, they can be identified with different physical actions and will make different problems easier, we may choose the representation for the scattering operator based on our needs. We shall briefly detail the three most useful representation now [24].

The **reflection-diagonal** representation is of particular use in the case of an impenetrable potential. In this case, two incoming particles with momenta p_1 and p_2 will interact with each other and thereby swap momentum but ultimately the relation $q_1 \lessgtr q_2$ will remain constant. Since A and B are the reflection and transmission coefficients respectively, this means that the S-matrix becomes diagonal. The second is the **transmission-diagonal** representation. It's effectively the same as for the reflection-diagonal except the rows of the S is changed such that in reflection-less potentials, the matrix is diagonalised. The final representation is that of the **transfer-matrix** where we want to describe particle motion between sectors. This is done by transferring amplitudes in the following way,

$$\begin{pmatrix} \mathcal{A}(21|12) \\ \mathcal{A}(21|21) \end{pmatrix} = \begin{pmatrix} -\frac{A}{B} & \frac{1}{B} \\ \frac{B^2-A^2}{B} & \frac{A}{B} \end{pmatrix} \begin{pmatrix} \mathcal{A}(12|12) \\ \mathcal{A}(12|21) \end{pmatrix}. \tag{3.28}$$

This combined with unitarity allows us to write,

$$T := \frac{1}{B(k)B(-k)} \begin{pmatrix} -A(k)B(-k) & B(-k) \\ B(k) & -A(-k)B(k) \end{pmatrix}, \tag{3.29}$$

the transfer matrix which moves a collection of amplitudes from one sector into another. It has the useful properties of being an involutory and traceless matrix. We should quickly note here that although they share the same form this transfer matrix is not identical to the likes of which that appears in studying the Ising model since in that case the partition function is given by $\mathcal{Z} = \mathrm{Tr}\left(T^N\right)$ for an N particle chain. Whereas under this, the trace would be null for N odd and just 2 for N even.

3.3.2. Bethe Wave Function

We now move on to the case of multibody scattering as an introduction to the Bethe Ansatz. Here we have an N particle system described by a wavefunction $\Psi(q_1, \ldots q_N)$ with time independent Schrödinger equation,

$$-\frac{\hbar^2}{2m}\sum_i \partial_i^2\Psi + \sum_{i\neq j} v(q_i - q_j)\Psi = E\Psi, \tag{3.30}$$

here the potential v is translational invariant such that the total momentum P is conserved. There is a notable complication that arises from treating $k > 2$ particle scattering events which we will discuss here.

Claim 4. *Factorised two body scattering interactions under a translational invariant potential obey non-diffractive momentum conservation where the set of momenta $\{p_i\}$ are mapped to a permutation of the same set in the final state. This is not the case for more complicated scattering events in general.*

Proof. For the two body case, we have conservation of energy and momentum. The question of what values of p_1 and p_2 are changed to after the scattering event then is a two dimensional problem - that is the momenta build a two dimensional space - which is constrained. The energy constraint, as expected, leads to the solutions living on a circle. Whilst the momentum conservation forces the solutions to live on a line as in Fig. 3.6. Therefore, the only valid solutions to the final momenta are given by the positive and negative permutation of the initial momenta. This is not the case in larger interactions where the outgoing asymptotic wave function is given by,

$$\Psi \sim \sum_{\tilde{p}_i} \mathcal{A}(\tilde{p}_i) e^{\frac{i}{\hbar}\sum_j \tilde{p}_j q_j} \delta\left(\sum_i \tilde{p}_i - P\right)\delta\left(\sum_i \epsilon(p_i) - E\right). \tag{3.31}$$

Which cannot be simplified any further for a general potential. Thus, the solutions to the momenta live on some plane in a three body scattering event and the scattering diffractive.

□

Suppose that we have a set of N pairwise commuting operators I_k where $H, P \in \{I_k\}$. Then the energy eigenfunctions will just be solutions to the eigenvalue equation, $I_m\Psi = h_m\Psi$. However, this could be just as difficult for particularly complicated potentials. So we assume further that the operators I_m are actually deformation of the free theory operators, $I_m^{(0)}$ since they are symmetric functions of the

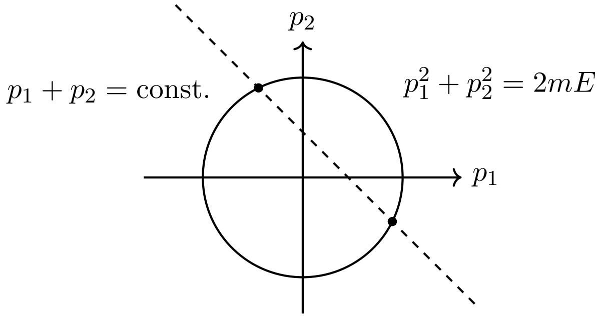

Figure 3.6. A diagram illustrating the conservation law origin of non-diffractive scattering.

momenta. We work in fundamental domain to begin with, that is to say where $q_i < q_{i+1}$.[7] Suppose we find the asymptotic solution with some ordering of momenta, $p_i < p_{i+1}$. In this case the full solution in the fundamental sector will just be collection of plane waves with the momentum and position indices matching and the full solution is given by,

$$\Psi(q_i) = \sum_{\tau \in \mathfrak{S}_N} \mathcal{A}(\tau) e^{\frac{i}{\hbar} \sum_i q_i p_{\tau(i)}}. \tag{3.32}$$

In general, we can divide the configuration space into $N!$ disconnected domains, each labelled by a permutation of the fundamental sector by a permutation of σ and called a σ-sector. Then the entire solution can be constructed by looking purely at the regions of each sector which is far from the boundary where no positions are close and so the potential term in small. We can then parametrise the asymptotic wave function as,

$$\Psi(q_i|\sigma) = \sum_{\tau \in \mathfrak{S}_N} \mathcal{A}(\sigma|\tau) e^{\frac{i}{\hbar} \sum_i q_{\sigma(i)} p_{\tau(i)}} \equiv \Psi(q_i) = \sum_{\tau \in \mathfrak{S}_N} \mathcal{A}(\sigma|\sigma\tau) e^{\frac{i}{\hbar} \sum_i q_i p_{\tau(i)}}. \tag{3.33}$$

This logic has left us with a remarkably simple expression for the wave function in the asymptotic region. We have just motivated the famed **Bethe wave function**. Which originally dates back to Bethe's work on ferromagnetic spin chains [20].

Supposing our potential is impenetrable and so we can intuit that its entirely reflection-based. As discussed earlier, this provides us with rationale to use the reflection representation to consider the scattering problem algebraically. The reflection representation is simple enough to understand, suppose the particle at position q_i scatters with the q_{i+1} particle where momenta undergo the transformation $p_{\tau(i)} \leftrightarrow p_{\tau(i+1)}$ then the colour of the particle is preserved. This is the most classically intuitive picture we can work in for these scattering events. We understand that when they are about to sink the 8-ball, sinking the white ball instead is an issue of skill and not of transmission. Defining the transposition operator, α_i which swaps the target's index $i \leftrightarrow i+1$, we can represent the change in the amplitudes after undergoing a scattering event in the right regular representation of the symmetric group, π, as,

$$\mathcal{A}(\sigma|\alpha_i\tau) = A(p_{\tau(i)}, p_{\tau(i+1)})\mathcal{A}(\sigma|\tau), \tag{3.34a}$$
$$\mathcal{A}(\sigma|\alpha_i\tau) = B(p_{\tau(i)}, p_{\tau(i+1)})(\pi(\alpha_j)\mathcal{A})(\sigma|\tau). \tag{3.34b}$$

[7] Further work can always be done by applying permutations. We note that here we will work with distinguishable particles for simplicity indexed by some quantum number. We choose to represent this by colour. One can always (anti) symmetrise the wave function when working with (fermions) bosons and we will discuss this briefly at the end of the section.

The former representing the action of a reflecting potential and the latter a transmitting event. If we define $\Phi(\tau) \coloneqq \{\mathcal{A}(\sigma|\tau),\ \sigma \in \mathfrak{S}_N\}$, a collection of all of the amplitudes in each different sector corresponding to one permutation of the momenta we can now start to describe the processes in a more concise algebraic fashion. This vectorisation allows us to rewrite (3.34) in the form,

$$\Phi(\alpha_i \tau) = Y_i(p_{\tau(i)}, p_{\tau(i+1)})\Phi(\tau), \tag{3.35}$$

where Y_i are Yang's scattering operators [2], [25],

$$Y_i(p_1, p_2) = A(p_1, p_2)\mathbb{1} + B(p_1, p_2)\pi(\alpha_i). \tag{3.36}$$

Then the we can build another set of consistency equations by identifying that squaring any transposition should yield the identity, coupled with the interpretation of Yang's operators as relating to evolutionary operators and thus must be unitary. This construction yields,

$$\Phi(\alpha_i \alpha_{i+1} \alpha_i) = \Phi(\alpha_{i+1} \alpha_i \alpha_{i+1}), \tag{3.37}$$

which should be setting off alarm bells in its similarity to (3.4). This is our first clue to the connection between factorised scattering and the Yang-Baxter equation. Indeed, we can define the scattering operator by considering the operator swapping any two indices and using it to promote Yang's operators,

$$S_{ij}(p_1, p_2) \coloneqq \pi(\alpha_{ij})Y_{ij} = B(p_1, p_2)\mathbb{1} + A(p_1, p_2)\pi(\alpha_{ij}). \tag{3.38}$$

This definition is quite profound yet also incredibly intuitive. Ultimately, it boils down to some chance of reflection related to A, in which nothing changes, and some chance of transmission related to B in which the two particles are swapped. A simple utilisation of the above consistency equations yield,

$$S_{12}(p_1, p_2)S_{13}(p_1, p_3)S_{23}(p_2, p_3) = S_{23}(p_2, p_3)S_{13}(p_1, p_3)S_{12}(p_1, p_2), \tag{3.39}$$

which is precisely (3.7) with the R-matrix and S-matrix identified.

Finally, we briefly consider the case of the particles having some internal degrees of freedom, perhaps spin. This allows the wave function decomposition,

$$\Psi \sim \underbrace{\psi(q_i)}_{\text{spatial}} \overbrace{\chi(s_j)}^{\text{spin}}. \tag{3.40}$$

We further assume that the particles are identical such that the Hamiltonian commutes with all transpositions but the transpositions do not mutually commute, as should be expected. This means there exists a sufficient degeneracy where each solution with a certain energy can be transformed into others by actions of the symmetry group. Therefore, we can label energy states by a symmetry group representation, λ. More precisely, the spatial and spin components are labelled individually by representations λ and $\bar{\lambda}$. Then to ensure the requisite overall symmetry properties of the solution, we see that for bosons $\lambda = \bar{\lambda}$ whereas for spin 1/2 fermions, the spin representation is conjugate to the spatial representation. A full discussion of this and a great explanation of the use cases of Young diagrams to this discussion can be found in [2].

3.3.3. Coordinate Bethe Ansatz

We now turn our attention to the first of two use cases of the Bethe ansatz out of the many found in modern physics. The coordinate Bethe ansatz, originally used by Bethe to describe ferromagnetic spin chains [20], we can now motivate and discuss it in order to provide a solid foundation for the algebraic Bethe ansatz to be discussed in Sec. 4.3. Let's suppose we're dealing with a system of N particles in a box of length L. As in elementary quantum mechanics, we must choose a boundary condition for our system. We identify both ends of the one-dimensional box with each other to gain periodic boundary conditions - meaning that the system effectively lives on a ring. We also suppose that the density N/L is sufficiently small so as to allow the necessary "asymptotic freedom" that our previous discussions relied upon. Under these assumptions, we write the Bethe wave function as,

$$\Psi(q_i) = \sum_{\sigma,\tau\in\mathfrak{S}_N} \mathcal{A}(\sigma|\tau) e^{\frac{i}{\hbar}\sum_{i=1}^{N} q_{\sigma(i)}p_{\tau(i)}}\, \Theta\left(q_{\sigma(1)} < q_{\sigma(2)} < \cdots < q_{\sigma(N)}\right), \tag{3.41}$$

where the coordinates q_i live on the line segment $[0, L)$ with a periodicity of L. That is to say,

$$\Psi(q_1, \dots, q_j = 0, \dots q_N) = \Psi(q_1, \dots, q_j = L, \dots q_N), \tag{3.42}$$

which can be expanded to the form,

$$\text{LHS} = \sum_{\sigma,\tau\in\mathfrak{S}_N} \mathcal{A}(\sigma|\tau) e^{\frac{i}{\hbar}\sum_{i=2}^{N} q_{\sigma(i)}p_{\tau(i)}}\, \Theta\left(q_{\sigma(2)} < \cdots < q_{\sigma(N)}\right)\, \delta(\sigma(1) = j), \tag{3.43}$$

$$\text{RHS} = \sum_{\sigma,\tau\in\mathfrak{S}_N} \mathcal{A}(\sigma|\tau) e^{\frac{i}{\hbar}\sum_{i=1}^{N} q_{\sigma(i)}p_{\tau(i)} + \frac{i}{\hbar}Lp_{\tau(N)}}\, \Theta\left(q_{\sigma(1)} < q_{\sigma(2)} < \cdots < q_{\sigma(N-1)}\right)\, \delta(\sigma(N) = j). \tag{3.44}$$

We have used in both cases that $q_j = 0$ for consistency purposes but on the RHS we have accounted for this with the pL term. Physically, we're just talking about choosing one particle to be at position 0 - wherever that may be - and describing it in two different domains. By defining ξ as the product of all neighbouring transpositions i.e. $\xi := \alpha_{N-1}\cdots\alpha_1$, and further redefining $\tau \mapsto \xi\tau$ we can express the RHS as,

$$\text{RHS} = \sum_{\sigma,\tau\in\mathfrak{S}_N} \mathcal{A}(\xi\sigma|\xi\tau) e^{\frac{i}{\hbar}p_{\xi\tau(N)}} e^{\frac{i}{\hbar}\sum_{i=2}^{N} q_{\sigma(i)}p_{\tau(i)}}\, \Theta\left(q_{\sigma(2)} < \cdots < q_{\sigma(N)}\right)\, \delta(\sigma(1) = j). \tag{3.45}$$

By noticing, then, that $\tau(\xi(N)) = \tau(1)$, we can identify LHS = RHS more easily and see,

$$\mathcal{A}(\sigma|\tau) = \mathcal{A}(\xi\sigma|\xi\tau) e^{\frac{i}{\hbar}p_{\tau(1)}L}. \tag{3.46}$$

This result can be thought of in two ways. The first is, essentially, as a consistency conditions. Viewing a particle at position 0 or L shouldn't make a difference and therefore for consistency, the amplitudes must obey this rule. This is how we have developed it thus far. Alternatively, we may think of physically dragging the particle around the ring and inspecting what has changed to its wave function, in principle similar to Aharanov-Bohm effect which is a duality which will be explored more later. Then by requiring the amplitude to remain ultimately unchanged by this action we arrive at this result. Indeed, by reintroducing the notation used in the previous section and utilising the form of the two-body S-matrix as in (3.38) we can express this result[8] as,

$$S_{j+1\ j} S_{j+2\ j} \cdots S_{N\ j} S_{1\ j} \cdots S_{j-1\ j} \Phi(e) = e^{\frac{i}{\hbar}p_j L}. \tag{3.47}$$

[8]Ultimately, a lot of these manipulations come down to just rewriting the permutations in a clever fashion. Or utilising the notation and directly using the right regular representation. Since these are in a sense trivial, we leave the workings-out to [2],

Now, we define the S-matrix product on the left hand side as the monodromy matrix T_j as well as the notational point $\Phi(e) \equiv \Phi$ coupled with setting $\hbar = 1$ for clarity we arrive at,

$$T_j \Phi = e^{ip_j L} \Phi. \tag{3.48}$$

Which are precisely the *Bethe-Yang* equations [25]. Here the duality we mentioned earlier might be slightly more apparent. The action of the mondoromy matrix, T_j, is to move the j^{th} particle around the ring. This movement resulting in a phase provides a potential comparison in the Berry phase [26]. The quintessential physical example of the Berry phase is the Aharanov-Bohm effect [27]. Which effectively stated that the motion of a charged particle through a region with non-zero vector potential would generate a phase acting on the particle's wave function even when in the absence of a magnetic field. This is reimagined in our formalism via the notion of *twisted* boundary conditions where $\Psi(q + L) = e^{i\phi}\Psi(q)$. However, it is important to point out the conceptual difference that here the phase comes from the existence of the particle in the position looped around whereas our phase coming from the action of the monodromy matrix is a phase originating from the motion itself.

One of the most natural manifestations of this principle, especially given its origin, is the spin chain. Here we let $V = \mathbb{C}^{2s+\mathbb{1}}$ a $2s + 1$ dimensional complex vector space for a spin s chain. Since an N-particle system then lives in $\mathscr{H} = V^{\otimes N}$, the representation of transpositions in the system then just become,

$$\pi_{ij} = \sum_{\alpha,\beta} \overbrace{\underbrace{\mathbb{1} \otimes \mathbb{1} \otimes \cdots \otimes E_{\alpha\beta}}_{i} \otimes \cdots \otimes E_{\beta\alpha}}^{j} \otimes \cdots \otimes 1, \tag{3.49}$$

where the matrices are in the i^{th} and j^{th} positions. This is nothing but the split Casimir operator as defined in (10).

Consider the following object,

$$T_a(p) = \pm \prod_{i=1}^{N} S_{ia}(p_i, p), \tag{3.50}$$

in which the sign indicates the behaviour of the S-matrix in the case of identical momenta relating to the (anti-)symmetric nature of the particles. This a indexes a separate copy of V called the auxiliary space. It is technically this operator which is called the monodromy matrix and V_a provides the space for particles to actually move through. Then taking the partial trace over the auxiliary space we attain a map between configuration spaces. This map, $\tau(p)$, is the transfer matrix. Then, by utilising the braiding property we can see that $\tau(p_j) = T_j$ as in (3.48). We can then note the utility of the mondromy matrices by the following propf.

Claim 5. *The operators T_i are commutative.*

Proof. Consider the left hand side of the RTT relation in (3.6) with the R-matrix identified as the S-matrix as discussed previously. Then we can write,

$$S_{ab}(p,q)T_b(q)T_a(p) = S_{ab}(p,q)S_{1b}(p_1,q)\cdots S_{Na}(p_N,p) = S_{ab}(p,q)S_{1b}(p_1,q)S_{1a}(p_1,p)\left(\cdots\right). \tag{3.51}$$

We move S_{1a} towards the front since it will commute with all other operators that act on different vector spaces. Then we can use the Yang-Baxter equation (3.7) to move the S_{ab} operator to the right.

Repetition of this process leaves S_{ab} entirely on the right and we retrieve the RTT relation. Then by inverting S_{ab} we get,

$$T_a T_b = S_{ab} T_b T_a S_{ab}^{-1}, \tag{3.52}$$

where the spectral parameters i.e. the auxiliary momenta have been omitted. Finally, by taking the trace we see that the cyclic symmetry simplifies the equation into,

$$\tau(p)\tau(q) = \tau(q)\tau(p), \tag{3.53}$$

which setting $p = p_i$ and $q = p_j$ we get,

$$T_i T_j = T_j T_i. \tag{3.54}$$

$\square$

4. The Hubbard Model

This text shall crescendo as we study a model of great relevance to condensed matter physics - in particular in its higher dimensional analogues seen in areas such as superconductivity. [28] In light of its importance, we shall discuss the Hamiltonian itself at length. We will then discuss its solution and finally formally introduce and study the Algebraic Bethe Ansatz via the Hubbard model as an example. Unfortunately, there is much relating to the physics of the model, in particular the thermodynamics and study of correlation functions, which is beyond the scope of this report. For more details on this we direct the reader to [4] as an excellent text that covers the subject in great breadth and depth. Indeed, the model can be used to describe aspects of electronic properties of solids with narrow bands, magnetism in metals such as iron and cobalt, the Mott insulating phase, as well as properties of high T_C cuprates.

The Hubbard model is a lattice model. That is to say, it describes a discretised real space on which there exists a number of electrons. We consider here just a one dimensional real space which greatly simplifies the discussion. Indeed, it is only in one dimension that the Hubbard model is integrable. More precisely, we only know that it is integrable in one dimension due to Shastry [29] and it is strongly implied that it should not be integrable for $d \geq 2$ but this has yet to be proven [30]. Lattice models, such as the Hubbard, are often well described by Bloch's theorem.

Theorem 3 (Bloch)**.** *The wave function of a particle restricted to a lattice, hence that with a periodic potential, is of the form [31],*

$$\psi(x) = e^{ikx}u(x), \tag{4.1}$$

where $u(x)$ is a function sharing the same periodicity as the potential.

Proof. Define the translation operator T_n which translates the system by n lattice steps. Since the potential is periodic in the lattice step then the translation operator commutes with the Hamiltonian. Then there exists a wave function which is an eigenstate of both the Hamiltonian and translation operator, which we denote $\psi(x)$. Since the translation operator is unitary, it's eigenvalues must be of the $\lambda = e^{i\theta}$. Then by applying the translation operator we get,

$$T_n\psi(x) = e^{in\theta}\psi(x) = \psi(x + na), \tag{4.2}$$

for a lattice spacing a. A solution to this equation is the required,

$$\psi(x) = e^{ikx}u(x). \tag{4.3}$$

□

In particular, we can relabel the wave function in (4.1) and promote it to a direct functional dependence on the quasi-momentum k and band index, $\varphi_{\alpha,k}$. This is begins a simplistic discussion amounting to a first quantised analysis. We shall only consider the second quantisation formalism.

4.1. The Hamiltonian

The central defining feature of any model is ultimately the source of its dynamics. Oft in field theories, in particular relativistic field theories, we deal with a Lagrangian but as the reader may have deduced from the Hamiltonian-centric description of integrability and this being a specifically non-relativistic problem we define here the Hamiltonian of the system.

$$H = -t \sum_{j=1}^{N} \sum_{\sigma=\uparrow,\downarrow} c^{\dagger}_{j,\sigma} c_{j+1,\sigma} + c^{\dagger}_{j+1,\sigma} c_{j,\sigma} + u \sum_{j=1}^{N} n_{j\uparrow} n_{j\downarrow}, \tag{4.4}$$

where $c^{\dagger}_{l,\sigma}$ is the creation operator of an electron at lattice site l with spin σ. Likewise $c_{l,\sigma}$ is the corresponding annihilation operator. These being fermionic means they obey the anti-commutation relation,

$$\{c_{i,\sigma}, c_{j,\sigma'}\} = \delta_{i,j}\delta_{\sigma,\sigma'}. \tag{4.5}$$

In future, we will utilise the notation,

$$\sum_{j,\sigma} := \sum_{j=1}^{N} \sum_{\sigma=\uparrow,\downarrow}, \tag{4.6}$$

and often neglect the explicit bounds of the sum for clarity. The reader should assume that we are summing over the full chain. A lot can be learned about the relevant physics merely from a thorough analysis of the Hamiltonian terms. The first term is called the *hopping term* with $t > 0$ the *hopping parameter*. [1] It is akin to a kinetic term in the Hamiltonian. Not in that it contains a time derivative but insofar as it describes motion. The hopping term consists of two building blocks, a creation term and an annihilation term with the same spin but offset by one lattice space. The first of the terms describes the motion of a spin σ particle from site $j+1 \to j$ whereas the second describes the reverse action. The second term, with coefficient u, is a bit more curious. The dynamics themselves depend immensely on the sign of u. If u is negative, this describes an attractive potential - seen by the system's desire to minimise the Hamiltonian. Due to Pauli's exclusion principle, the $n_{j,\sigma}$ terms can only be one or zero. If u is negative then the manner of minimising the Hamiltonian necessarily includes grouping two electrons at lattice sites so as to ensure that the attractive term contributes. This notion of an attractive interaction is not necessarily unheard of, as it is a feature of BCS superconductivity, but the nature of which it appears is due to renormalisation of the repulsive interaction. So the bare attractive interaction isn't too physically relevant. What is far more relevant is the repulsive regime where $u > 0$. In this case, the electrons ought to spread out such that the number density term never contributes. This doesn't mean much for lowly filled systems where the electrons are free to move around without interacting a whole lot. Consider a very high u system and we would expect to see electrons bounce around the chain, reflecting off of each other when scattering.

However, the most interesting physics here occurs when we reach **half-filling**. In this case, we have a system with N electrons in a chain of length $L = N$. If u is sufficiently low, we might expect to

[1] We often set $t = 1$ and deal only in units of u.

see charge transfer through the system i.e. a current, hence the system is still conductive. But if u is large then if it not energetically feasible to have a configuration where electrons can double up. Therefore, charge cannot move and the system is an insulator. This manner of becoming an insulator is exotic compared to the usual form of the phase which occurs from a large band gap between an valence band a conductive band. The insulating property comes not from this band theory but rather the electron-electron repulsion. The system is hence called a **Mott insulator**.

We now switch our focus from an analysis of the dynamics, to an attempt to utilise the Hamiltonian to gain insight about the solutions a priori. The first things we may notice is that the system clearly has a Hilbert space of dimensions 4^L and some easily found conserved quantities - namely the number of up and down spins, $n_\uparrow$ and $n_\downarrow$ respectively, and the total spin in the z direction, S^z. We can also define the unitary shift operator[2] which moves the position 1 to n as,

$$U_n := P_{n-1,n} \cdots P_{2,3} P_{1,2}, \qquad U_n^\dagger := P_{1,2} P_{2,3} \cdots P_{n-1,n} \implies U_n^\dagger U_n = 1. \tag{4.7}$$

Having defined this shift operator, it is natural to attempt to define a momentum operator Π such that,

$$U = e^{i\Pi}, \qquad \Pi = \Pi^\dagger, \qquad [H, \Pi] = 0. \tag{4.8}$$

Naïvely, one might choose the momentum by first Fourier transforming to momentum space, then attaching the number density at momentum k with the momentum k and sum over all "momenta". Thus resulting in,

$$\tilde{\Pi} = \phi \sum_k k\, c_k^\dagger c_k. \tag{4.9}$$

However, one need only consider the translation $\phi \to \phi + 2\pi$ to see the issue. In particular, the shift operator then should be transform in an expected but the momentum is not - a poor sign. This natural, physical construction for the momentum seems to echo that of a *mechanical* momenta as in classical mechanics. Indeed, it can be shown that $\tilde{\Pi}$ is also not a conserved quantity, giving further fuel for our analogy. However, we are looking from the *canonical* momentum. The actual derivation of the momentum operator [4] is not illuminating but results in,

$$\Pi = \phi \sum_{m=1}^{L-1} \frac{1}{2} + \frac{U^m}{e^{-im\phi} - 1}. \tag{4.10}$$

This result is found to act on a basis state as,

$$\Pi \left|\mathbf{k}, \mathbf{a}\right\rangle = \Big(\phi(k_1 + k_2 + \cdots + k_n) \mod 2\pi\Big) \left|\mathbf{k}, \mathbf{a}\right\rangle. \tag{4.11}$$

Furthermore, we can utilise some more interesting discrete symmetries. First and foremost, the spin reversal operator,

$$J^{(s)} := \prod_{j=1}^{L} P_{j\uparrow, j\downarrow}. \tag{4.12}$$

[2] Technically, we define it here for a spinless fermion. The spin generalisation is trivial and requires just attaching a spin index.

Which will feature later. The particle-hole transformation under the assumption that there are an even number of lattice sites[3],

$$J_a^{sh} = (c_{L,a}^\dagger - c_{L,a})(c_{L-1,a}^\dagger + c_{L-1,a})\cdots(c_{2,a}^\dagger - c_{2,a})(c_{1,a}^\dagger + c_{1,a}). \tag{4.13}$$

It's clear to see that the particle-hole transformation commutes with the creation and annihilation operators of the opposite spin species. Furthermore, we can ask how the annihilation operator transforms under the adjoint action,

$$J_\downarrow^{sh} c_{a,\downarrow} (J_\downarrow^{sh})^\dagger = (c_{L,\downarrow}^\dagger - c_{L,\downarrow})\cdots(c_{a,\downarrow}^\dagger - c_{a,\downarrow})c_{a,\downarrow}\cdots(c_{1,\downarrow}^\dagger + c_{1,\downarrow})(c_{1,\downarrow}^\dagger + c_{1,\downarrow})(-c_{2,\downarrow}^\dagger + c_{2,\downarrow})\cdots, \tag{4.14}$$

which simplifies down to merely just $(-1)^a c_{a,\downarrow}^\dagger c_{a,\downarrow} c_{a,\downarrow}^\dagger = (-1)^a c_{a,\downarrow}^\dagger$. Thus, the hopping term in the Hamiltonian is invariant and the action on the number density operators is effectively just to flip the order of creation and annihilation operators. Reversing this back amounts to sending $u \to -u$. This action, also known as the *Shiba transformation* is equivalent to the transformation from the repulsive to the attractive regime and vice versa.

The next point of discussion will develop into a very curious example of a hidden symmetry. As is plain from the $U(1)$ symmetry characteristic of quantum mechanical systems, the total number of particles is conserved. The lack of an explicit mention of spin in the Hamiltonian means that the total spin vector is also conserved. So we have currently four independent conserved quantities. There is, lurking in the shadows, another symmetry of the system akin to spin. To see this, we discuss briefly the solution of the system via the Bethe Ansatz.

4.2. Coordinate Bethe Ansatz Solution of the Hubbard Model

The simplest solutions of the Hubbard model requires the use of the coordinate representation of the Hamiltonian and hence is based on the Coordinate Bethe Ansatz. We can easily convert the hopping term into shift operators, $\Delta_j^\pm$ which act on the wave function as,

$$\Delta_j^\pm \psi(\mathbf{x} \pm \mathbf{e}^j; \mathbf{a}), \tag{4.15}$$

where $j = 1, 2, \ldots, N$. Then the Hamiltonian can be rewritten, together with a reparametrisation $u \to 4u$ just to match the literature,

$$\tilde{H} = -\sum_{j=1}^{N} \left(\Delta_j^+ + \Delta_j^-\right) + 4u \sum_{i<j} \delta_{x_i, x_j}. \tag{4.16}$$

We obviously require the wave function to be a solution to the difference energy eigen-equation but we also require the solution to obey periodic boundary conditions with respect to the shift operators.

[3]Although as with all issues of the number of sites and boundary conditions this will disappear in the thermodynamic limit.

4.2.1. Two Particle System

We first attempt to find a solution to the simple case where there are two particles. As per usual, we break the wave function into its centre of mass and relative motion components, $\psi(x_1, x_2; a_1, a_2) = f(m)g(n)$ with $m = x_1 + x_2$ and $n = x_1 - x_2$. The solutions to the system in this case take the form,[4]

$$\begin{aligned} f(m+1) + f(m-1) &= Cf(m), \\ C\Big(g(n+1) + g(n-1)\Big) &= \Big(4u\delta_{n,0} - E\Big)g(n). \end{aligned} \tag{4.17}$$

From the translational symmetry we should require the solution to be an eigenfunction of the shift operator, $U \coloneqq \prod_i \Delta_i^+$. This results in another constraining,

$$f(m+2) = \omega f(m), \tag{4.18}$$

with ω being the eigenvalue of the shift operator. Defining w such that $w + 1/w = C$ we get a solution to f of the form,

$$f(m) = A^+ w^m + A^- w^{-m}. \tag{4.19}$$

The latter condition ensures that $w^2 = \omega$. We assume that only the positive power term contributes as so $f(m) = w^m$ since the overall factor can be absorbed.

The function which describes the relative motion is a bit more complicated to discuss. As one would notice, there is a particularly interesting point for which $C = 0$ and then w is defined with solution $\pm i$ and corresponds to the degenerate case. Therefore, for the time being, we deal purely with the non-degenerate case $C \neq 0$. The other complicating matter is that of spin. Compiling all of the spin degrees of freedom into $g(n)$ means that we should expect the solution to have spinor indices too. Indeed, we find that in general,

$$g(n) = \begin{cases} A^{-+} z^n - A^{--} z^{-n}, & n > 0 \\ A^{++} z^n - A^{+-} z^{-n}, & n < 0, \end{cases} \tag{4.20}$$

under the condition that the sum of alternating spinor indices is equal to the sum of the amplitudes with the same indices. This being the validity of the function at $n = 0$. The Hamiltonian condition then yields two solutions for z in terms of w,

$$wx = e^{ik_1} \quad \text{and} \quad \frac{w}{z} = e^{ik_2}, \quad \text{for some } k_1, k_2 \in \mathbb{C}. \tag{4.21}$$

It is now useful to define a few quantities. Firstly, $s_j \coloneqq \sin(k_j)$ together with the permutation matrix Π which interchanges spinor indices. Lastly, we define the *Yang operator*,

$$Y(\lambda) \coloneqq \frac{2iu + \lambda\Pi}{2iu + \lambda}. \tag{4.22}$$

The same Hamiltonian condition as earlier, coupled with the definitions of k_k, provides us with an incredibly insightful way of rewriting the form of the wave function,

$$\psi(x_1, x_2) = \begin{cases} A^{-+} e^{i(k_1 x_1 + k_2 x_2)} - Y(s_1 - s_2) A^{-+} e^{i(k_1 x_2 + k_2 x_1)}, & x_1 \leq x_2 \\ Y(s_1 - s_2)\Pi A^{-+} e^{i(k_1 x_1 + k_2 x_2)} - \Pi A^{-+} e^{i(k_1 x_2 + k_2 x_1)}, & x_2 > x_1. \end{cases} \tag{4.23}$$

[4] We collect all spin degrees of freedom to the function $g(n)$.

We now further break up the wave function into the spin singlet and spin triplet states, recalling, of course that the electron pair will decompose as $\frac{1}{2} \otimes \frac{1}{2} \simeq 0 \oplus 1$ with (anti-)symmetric behaviour respectively. The overall symmetry properties of the wave function, we condense the solution to the following forms,

$$\begin{aligned} \psi(x_1, x_2; a_1, a_2) = \phi_a \cdot &\begin{cases} e^{i(k_1x_1+k_2x_2)} - Y(s_1 - s_2)e^{i(k_1x_2+k_2x_1)}, & x_1 \leq x_2 \\ Y(s_1 - s_2)e^{i(k_1x_1+k_2x_2)} + e^{i(k_1x_2+k_2x_1)}, & x_2 > x_1 \end{cases} \qquad \text{for the singlet and} \\ \psi(x_1, x_2;a_1, a_2) = \phi_s &\Big(e^{i(k_1x_1+k_2x_2)} + e^{i(k_1x_2+k_2x_1}\Big) \qquad \text{for the spin triplet.} \end{aligned} \tag{4.24}$$

As mentioned previously, we now turn our attention to the degenerate $C = 0$ case. This yields the relative motion Hamiltonian eigen equation,

$$\Big(4u\delta_{x_1,x_2} - E\Big)\, g(n) = 0. \tag{4.25}$$

There are two energy values for which $g(n)$ has an interesting solution, $E = 0$ and $E = 4u$. The former case isn't quite as exotic. Indeed, it ultimately results in the same procedure - except for the relaxation of a previously unmentioned constraint. Where previously, we needed to ensure $k_1 + k_2 \neq \pi \mod 2\pi$ now we do not. When $E = 4u$ for this to be true we need $g(n) = A\delta_{n,0}$ where A is anti-symmetric under permutation. Without loss of generality we say that $g(n) = \phi_a\delta_{x_1,x_2}$. Doing so we arrive at a bound state localised in the centre of mass frame,

$$\psi(x_1, x_2; a_1, a_2) = \phi_a(-1)^{x_1}\delta_{x_1,x_2}. \tag{4.26}$$

If we put the two particle system on an infinite interval we can see that the spin triplet states are always bounded for purely real quasi-momentum, k. The discussion becomes a bit more nuanced for the case of k_j with non-zero imaginary part. Indeed, for the spin singlet case we have a bounded solution if and only if $s_1 - s_2 = 2iu$. Here the wave function nicely factorises into,

$$\psi = \phi_a \cdot e^{\frac{i}{2}(k_1+k_2)(x_1+x_2)} e^{-\frac{i}{2}(k_1-k_2)|x_1-x_2|}. \tag{4.27}$$

This is clearly bounded only if the imaginary part of $k_1 + k_2$ is null whilst the imaginary part of the difference is non-positive. It turns out that we can we can write the relative quasi-momentum in terms of the total quasi-momentum. The manner of which we can attain this relation implies that the bound state is possible even for a repulsive potential à la Cooper pairs. This property, interestingly enough, is due to the lattice effects and is lost in the continuum limit.

Another question the reader may have it due to the topology the system exists on. Putting the system on a ring proves to be useful in one respect and simply curious in the other. The former concerns the statement allowing us to impose a sufficient condition on how the Yang operator, or more explicitly the S-matrix $X(s_1 - s_2) := Y(s_1 - s_2)\Pi$, acts on the spinor components A^{-+}. The identification $A^{-+} = \phi_a$ satisfies this condition if the quasi-momenta satisfy,

$$e^{ik_1L} = \frac{s_1 - s_2 + 2iu}{s_1 - s_2 - 2iu}, \qquad e^{ik_2L} = \frac{s_2 - s_1 + 2iu}{s_2 - s_1 - 2iu}. \tag{4.28}$$

The latter is effectively the realisation of the degeneracy between the spin symmetric and anti-symmetric states. This degeneracy is lifted and the underlying symmetry, a *Yangian symmetry* [4] is broken, under periodic boundary conditions.

4.2.2. Many Particle System

We now turn our attention to the many particle case. In general, the Bethe Ansatz states depend on two quantum numbers: a charge momentum k_j as in the previous subsection and a spin rapidity λ_i, where $j = 1, 2, \ldots, N$ and $i = 1, 2, \ldots, M$. Here N is the total number of particles and M is the total number of down spin electrons. Both of which are conserved quantities. These quantum numbers satisfy a set of coupled algebraic equations due to the periodicity of the system called the **Lieb-Wu** equations,

$$\begin{gathered} e^{ik_jL} = \prod_{l=1}^{M} \frac{\lambda_l - s_j - iu}{\lambda_l - s_j + iu}, \qquad i = 1, \ldots, N, \\ \prod_{j=1}^{N} \frac{\lambda_l - s_j - iu}{\lambda_l - s_j + iu} = \prod_{\substack{m=1, \\ m \neq l}}^{M} \frac{\lambda_l - \lambda_m - 2iu}{\lambda_l - \lambda_m + 2iu}, \qquad l = 1, \ldots, M. \end{gathered} \tag{4.29}$$

In the scope of this work we only consider finite quantum numbers with the condition $2M \leq N \leq L$. That is to say, there can be at most one electron per lattice site of which at most $N/2$ can be spin down. The former allows us to discuss the conducting phase up to the Mott insulating phase as mentioned previously. The rationale behind the latter will become apparent over the remainder of the discussion of the coordinate Bethe Ansatz. However, it amounts to ensuring that we do not overcount. These solutions are called regular and vitally can be further constructed into a complete set. This completeness is key to the utility of the Bethe Ansatz itself. A more full discussion of this can be found in Chapter 4 of [4] but is out of the scope of this document.

The solutions look something along the lines of,

$$|\psi_{\mathbf{k},\boldsymbol{\lambda}}\rangle = \frac{1}{N!} \sum_{x_i=1}^{L} \sum_{a_j=\uparrow,\downarrow} \psi(\mathbf{x}; \mathbf{a} \,|\, \mathbf{k}, \boldsymbol{\lambda}) \,|\mathbf{x}, \mathbf{a}\rangle \,, \tag{4.30}$$

where the sum is taken over all i and j. The exact form of the wave functions are not relevant to our discussion but again can be found in Appendix 3.B in [4]. There is associated to each solution, a sector $\sigma \in \mathfrak{S}^N$ such that $1 \leq \cdots \leq x_{\sigma(i)} \leq x_{\sigma(i+1)} \leq \cdots \leq L$, of which there are $N!$. This should remind the reader of the coordinate Bethe Ansatz wave functions as in (3.41). Through this method we find the eigenvalues of the energy and momenta corresponding to these states,

$$E = -2\sum_{j=1}^{N} \cos(k_j) + u(L - 2N), \qquad P = \sum_{j=1}^{N} k_j \mod 2\pi. \tag{4.31}$$

We now move on to discussing the symmetry properties of the states. The first of which are those relating to permutation i.e. the symmetric group. These, put simply, are represented by the action on the wave function [4],

$$\begin{aligned} \psi(\mathbf{x}\sigma; \mathbf{a}\sigma \,|\, \mathbf{k}, \boldsymbol{\lambda}) &= \operatorname{sgn}(\sigma)\psi(\mathbf{x}; \mathbf{a} \,|\, \mathbf{k}, \boldsymbol{\lambda}) \quad \sigma \in \mathfrak{S}^N, \\ \psi(\mathbf{x}; \mathbf{a} \,|\, \mathbf{k}\sigma, \boldsymbol{\lambda}) &= \operatorname{sgn}(\sigma)\psi(\mathbf{x}; \mathbf{a} \,|\, \mathbf{k}, \boldsymbol{\lambda}) \quad \sigma \in \mathfrak{S}^N, \\ \psi(\mathbf{x}; \mathbf{a} \,|\, \mathbf{k}, \boldsymbol{\lambda}\sigma) &= \psi(\mathbf{x}; \mathbf{a} \,|\, \mathbf{k}, \boldsymbol{\lambda}) \quad \sigma \in \mathfrak{S}^N. \end{aligned} \tag{4.32}$$

Doing so, we see easily that the electrons themselves, defined by $\mathbf{x}$ and $\mathbf{a}$ i.e. their position and spin states, are fermionic - as expected. Furthermore, we may see that the quasi-particles that carry the charge momenta are fermionic and that the quasi-particles to which the spin rapidities define are bosonic. The notion of some quasi-particles operating in the system being bosonic whilst other acting as fermions is an observation which may seem very attractive right now. However, the story is a bit more nuanced than that since it is ultimately a description of how the quantum numbers enter the wave function. It is, however, a good indicator of some interesting mixing of fermionic and bosonic properties relevant to the model. This notion is central to our discussion of the algebraic Bethe Ansatz via the a grading on the relevant algebra.

Finally, we have arrived at the aforementioned hidden symmetry. Or at least a taste-test of it, though its utility is not manifestly apparent just yet. Indeed, to see it arise we first must set L to an even number - ultimately this doesn't matter in the thermodynamic limit where boundary conditions and the precise parity of the chain are irrelevant. It turns out that [4] the action of the spin weight increasing operator, S^+, annihilates the Bethe states i.e. $S^+ |\psi\rangle = 0$ and thus these are the highest weight states in a particular representation. The details of which can be found by acting S^z on the state,

$$S^z |\psi\rangle = \frac{1}{2}(N - 2M) |\psi\rangle . \tag{4.33}$$

We can further utilise this by applying a Shiba transformation to the relevant operators therefore defining a new algebra,

$$\mathrm{Ad}_{J^{sh}_{\downarrow}} S^{\pm} =: -\eta^{\pm}, \qquad \mathrm{Ad}_{J^{sh}_{\downarrow}} S^{\pm} =: \eta^{z}. \tag{4.34}$$

It's clear to see that the η algebra follows precisely the same commutation relations as the spin operators and hence are a representation of $\mathfrak{su}(2)$ and are conserved. Identically, the Bethe states are also the lowest weight states of the η algebra with weight $\frac{1}{2}(N - L)$. Thus, we can construct a degenerate multiplet of eiegenstates with dimension $(N - 2M + 1)(L - N + 1)$.

The Bethe states being the lowest weight states in the η algebra but the highest in the spin algebra is a bit annoying. Often then we actually define the ζ algebra to be identical[5] except for $\zeta^z = -S^z$. Now the Bethe states are the lowest weight states in both cases. We can therefore "fill out" the rest of the representation by just acting with the raising operator on these states,

$$|\psi_{\mathbf{k},\boldsymbol{\lambda};\alpha,\beta}\rangle := (\zeta^{\dagger})^{\alpha}(\eta^{+})\beta\, |\psi_{\mathbf{k},\boldsymbol{\lambda}}\rangle . \tag{4.35}$$

We might think that all is well and we have found that the total symmetry group of the system is found by exponentiating $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. There is, however, a subtlety here concerning the available eigenvalues of the operator $\eta^z + \zeta^z$,

$$(\eta^z + \zeta^z) |\psi\rangle = \frac{1}{2}(2M - L) |\psi\rangle . \tag{4.36}$$

For L even this is clearly an integer. Hence, we must require that either the ζ^z and η^z eigenvalues are both integers or both half-integers since any mixing will contradict this finding. Therefore, there exists a further $\mathbb{Z}_2$ factorisation on the group level. That is to say, the symmetry group of the system is actually,

$$\frac{SU(2) \times SU(2)}{\mathbb{Z}_2} \simeq SO(4). \tag{4.37}$$

[5] In the literature the notation is actually more similar to the Heisenberg algebra insofar as $S^+ \mapsto \zeta^\dagger$ and $S^- \mapsto \zeta$.

This is our hidden symmetry. [6] These further explored states are complete. That is to say, the Bethe Ansatz coupled with this $SO(4)$ symmetry yield all eigenstates of the Hubbard Hamiltonian.

4.3. Algebraic Bethe Ansatz

We now conclude our discussion of the Hubbard model with a dual explanation via example of the Algebraic Bethe Ansatz. Before diving into the Hubbard model we first must define what a fundamental model is. As per usual and explained in more detail in Chapter 3 we can define the transfer matrix in the usual way with $L(\mu, v_i)$ which satisfies (3.6) i.e.

$$T(\mu) = L_L(\mu, v_L) \cdots L_1(\mu, v_1). \tag{4.38}$$

This defines the fundamental model associated with a given R-matrix and is a representation of the Yang-Baxter algebra.

Definition 15 (Regular R-Matrix)**.** *An R-matrix is called regular if there exists spectral parameters, λ_0 and μ_0, such that the components of the R-matrix has value,*

$$R(\lambda_0, \mu_0)^{\alpha\beta}_{\gamma\delta} = \delta^{\alpha}_{\delta}\delta^{\beta}_{\gamma}. \tag{4.39}$$

Following this, the L-matrix has value $L^{\alpha}_{j\,\beta}(\lambda_0, \beta_0)$. Again, recalling our previous definitions we may take the trace over the monodromy matrix to acquire the transfer matrix. This yields,

$$t(\lambda_0) = P_{12}P_{23} \cdots P_{L-1\,L} = U, \tag{4.40}$$

the rightward shift operator.

We recall from the definition of the transfer matrix that its utility is derived through the generation of an infinite set of conserved charges. This is done via Laurent expansion of the spectral parameters in which the set of coefficient operators form the generators of corresponding conserved quantities. However, this property is generalisable. In particular, any differentiable function of t is too able to generate these conserved quantities. The most apparent example of this we denote the **generating function**,

$$\tau(\lambda) := \ln\Big(U^{-1}t(\lambda)\Big) = (\lambda - \lambda_0)U^{-1}t'(\lambda_0) + \mathcal{O}(\lambda^2). \tag{4.41}$$

This linear term we define to be the Hamiltonian[7] $H = \sum_{j=1}^{L} H_{j-1,j}$. This sum over Hamiltonian densities can be found in terms of the twisted R-matrix,

$$H_{j-1,j} = \partial_\lambda \check{R}_{j-1,j}\Big|_{\lambda=\lambda_0}. \tag{4.42}$$

Turning our attention to the monodromy matrix itself, we may ask how it acts on varying states. Indeed, this question begs another - what are the states we are acting on? We do this constructively as

[6] Discussion of a hidden symmetry and $SO(4)$ might ring some bells for the reader. Indeed, it is precisely the same group that pertains to the hidden symmetry in the Hydrogen Atom [32] and equivalently Kepler's Problem. Although to the best of our ability we could not find a direct connection between these scenarios. It seems like a mere coincidence.

[7] We recall our implementation of periodic boundary conditions.

in second quantised quantum mechanics by creating a vacuum state to which we can create particles.[8] We consider the elements of the monodromy matrix to act like particle creation and annihilation operators on this pseudo-vacuum, $|\Omega\rangle$. In fact, from all known example the elements of the monodromy matrix can be arranged such that it acts as an upper triangular matrix on $|\Omega\rangle$. The canonical example of this construction seems to be the XXX-Model [2], [4]. Here the monodromy matrix looks like,

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \tag{4.43}$$

which acts on the pseudo-vacuum as,

$$A(\lambda)\,|\Omega\rangle = a(\lambda)\,|\Omega\rangle\,, \quad D(\lambda)\,|\Omega\rangle = d(\lambda)\,|\Omega\rangle\,, \quad C(\lambda)\,|\Omega\rangle = 0. \tag{4.44}$$

The functions $a(\lambda)$ and $d(\lambda)$ define parameters of a generalised model. The use of this to characterise the representation is in analogy to the manner of which the highest weight vector defines a representation of a Lie Algebra. Furthermore, we use the top left element $B(\lambda)$ as a creation operator to build the state,

$$|\mu_1 \cdots \mu_M\rangle \coloneqq B(\mu_1)\cdots B(\mu_M)\,|\Omega\rangle\,. \tag{4.45}$$

The goal, of course, is to find states which hold all conserved quantities simultaneously. The object which holds this information is the transfer matrix and thus we must diagonalise and hence we encounter the eigenvalue problem,

$$A(\lambda) + D(\lambda) =: t(\lambda)\,|\psi\rangle = \Lambda(\lambda)\,|\psi\rangle\,. \tag{4.46}$$

We can use (3.6) to construct 16 quadratic relations including the notable commutation of the creation operators. Here we interpret the R-matrix as a scattering matrix and decompose $R(\lambda,\mu) = b(\lambda - \mu)\mathrm{id} + c(\lambda-\mu)P$, i.e. the functions b and c act like transmission and reflection coefficients. Doing so we can find the eigenvalue,

$$\Lambda(\lambda) = a(\lambda)\prod_{k=1}^{M}\frac{1}{c(\mu_k-\lambda)} + d(\lambda)\prod_{k=1}^{M}\frac{1}{c(\lambda-\mu_k)}. \tag{4.47}$$

To obtain this result we require,

$$\frac{d(\mu_k)}{a(\mu_k)} = \prod_{\substack{i=1\\ i\neq k}}^{M}\frac{c(\mu_k-\mu_i)}{c(\mu_i-\mu_k)}, \tag{4.48}$$

which are nothing other than the Bethe equations. This property ensures [4] that the states generated by the $B(\mu)$ operators are indeed eigenstates and hence relevant to our physical discussion. We can then just use the prior definition (4.42) and hence the Hamiltonian to find the energy eigenvalues of the system.

[8]This assumption does not always hold. In systems like the Toda chain we must instead use techniques including the method of seperation of variables to solve it instead.

4.3.1. Shastry's R-Matrix

The construction of an R-matrix for the Hubbard model was long known as a tremendously difficult problem. One pertinent reason for this is due to an argument from Reshetikhin that the Hubbard model's R-matrix could not be formulated in a difference form. This property being peculiar in relation to other systems. Another issue is that there are not just spin degrees of freedom, as in the spin chain, but also fermionic degrees of freedom. This is circumvented by way of a **Jordan-Wigner transformation** which we will detail later. This allows us to map a fermionic system onto a spin system while preserving the proper anti-commutation relations. An R-matrix for this transformed model was obtained by Shastry [29], [33] in 1986. We introduce this by somewhat following the discussion [34] and the general formalism of graded spaces outlined in [35].

We begin by introducing the $\mathfrak{su}(d)$-XX models as building blocks of the construction of Shastry's R-matrix. We consider a special case of the XYZ model with $J_x = J_y = \frac{J}{2}$ and $J_z = 0$ with periodic boundary conditions. The Hamiltonian of this system can be written as,

$$H_{XX} = \frac{J}{2}\sum_{j=1}^{N}\left(\sigma^x_{j-1}\sigma^x_j + \sigma^y_{j-1}\sigma^y_j\right) = J\sum_{j=1}^{N}\left(\sigma^+_{j-1}\sigma^-_j + \sigma^-_{j-1}\sigma^+_j\right), \tag{4.49}$$

which should start to look not too dissimilar to the Hubbard hamiltonian's hopping term. Since $\sigma^{\pm}$ are the generators of a fermionic anti-commutation relation algebra for a single site this makes sense. However, they fail to generate the multi-site fermionic relations. This is fixed by means of the Jordan-Wigner transformation which allows us to faithfully represent fermionic creation and annihilation operators with $\mathfrak{sl}(2)$ elements,

$$\begin{aligned} c_j &= \mathbb{1}_2^{\otimes(j-1)} \otimes \sigma^+ \otimes (\sigma^z)^{\otimes(L-j)} \\ c_j^\dagger &= \mathbb{1}_2^{\otimes(j-1)} \otimes \sigma^- \otimes (\sigma^z)^{\otimes(L-j)} \end{aligned}. \tag{4.50}$$

This transformation manifestly identifies the relation between a spin system and a spinless, fermionic tight-binding model.

Since we are working with algebras anyway, let us now take the time to define a particularly relevant algebra to the following discussion.

Definition 16 (Free Fermion Algebra)**.** *Let $\mathcal{U}$ be a unital, associative algebra whose elements, A_i, satisfy the following conditions,*

$$\begin{aligned} A_iA_{i+1}A_i =& A_{i+1}A_iA_{i+1} = 0 \\ A_iA_j = A_jA_i&, \quad \textit{for } |i-j| > 1 \\ A_i^3 &= A_i \\ \{A_i^2, A_{i\pm1}\} &= A_{i\pm1}. \end{aligned} \tag{4.51}$$

Furthermore, we impose periodic boundary conditions on these generators. The algebra, $\mathcal{U}$, is then called the Free Fermion Algebra [34]. The first of these means that elements of the algebra satisfy the braid relation.

We can use this free fermion algebra to define the representation,

$$A_i = Q_{i,i+1} = Q^{\alpha\gamma}_{\beta\delta}\, e_i{}^{\beta}{}_{\alpha}\, e_{i+1}{}^{\delta}{}_{\gamma}, \tag{4.52}$$

through which we may define, using the notation of [4], $P^{(1)} = Q^2$ and $P^{(2)} = \mathrm{id} - P^{(1)}$. Finally, we may construct the twisted R-matrix,

$$\check{R}(\lambda) = P^{(1)} + P^{(2)} \cos(\lambda) + Q \sin(\lambda). \tag{4.53}$$

It should be noted that this R-matrix is technically not unitary although division by $\cos(\lambda)$ suffices to reinstitute unitarity. Furthermore, the Hamiltonian density in this case is just $\mathcal{H}_{j-1} = Q_{j-1,j} = A_{j-1}$. The explicit form of $\check{R}$ is vital to constructing Shastry's R-matrix later on. However, for the sake of space we have chosen to omit all of the forms of these objects for brevity. We will outline the steps but for a more comprehensive discussion and the elements of all objects we direct the reader to [4].

Definition 17 (Conjugation Matrix)**.** *The conjugation matrix $C \in \mathrm{End}(\mathbb{C}^d)$ is defined by the following properties,*

$$C^2 = \mathbb{1}_d \tag{4.54}$$

$$\{C_i, Q_{12}\} = 0, \quad i = 1, 2 \tag{4.55}$$

$$C_1 Q_{12} = Q_{12} C_2 \tag{4.56}$$

$$Q_{12}^2 = \frac{1}{2}\left(\mathbb{1}_{d^2} - C_1 C_2\right). \tag{4.57}$$

Here $Q \in \mathrm{End}(\mathbb{C} \otimes \mathbb{C})$ is a representation of the free fermion algebra defined above.

The action of this conjugation matrix on $\check{R}$ as in (4.53) is to send $\lambda \mapsto -\lambda$. Doing so we can obtained the decorated Yang-Baxter equation,

$$R_{12}(\lambda + \mu) C_1 R_{13}(\lambda) R_{23}(\mu) = R_{23}(\mu) R_{13}(\lambda) C_1 R_{12}(\lambda + \mu). \tag{4.58}$$

With these preliminary aspects covered we move on to outlining Shastry's construction itself. Put simply, his procedure is as follows. First, we construct a higher conserved charge of the Hubbard model, I. That is to say, an operator which is independent from but commutes with the Hamiltonian. Secondly, by means of the Jordan-Wigner transformation (4.50) we obtain spin chain operators, H^s and I^s. We then guess the L-matrix such that the series of conserved quantities generated by the generating function has as its linear term, the Hamiltonian, and as its quadratic, I^s. Finally, we then construct an R-matrix such that his this L-matrix is a solution of T in (3.6).

Truth be told, the entire procedure is a bit laborious and so we only focus on this last step. We go about solving for this R-matrix by composing two XX model R-matrices. So consider this system of non-interacting XX models. We define a set of almost-R-matrices and transposition operators,

$$\begin{aligned} \check{r}_\uparrow(\lambda) &:= \check{R}^{\alpha\gamma}_{\beta\delta}(\lambda) e^\beta_\alpha \otimes \mathbb{1}_d \otimes e^\gamma_\delta \otimes \mathbb{1}_d \\ \check{r}_\downarrow(\lambda) &:= \check{R}^{\alpha\gamma}_{\beta\delta}(\lambda)\, \mathbb{1}_d \otimes e^\beta_\alpha \otimes \mathbb{1}_d \otimes e^\gamma_\delta \end{aligned}, \qquad \begin{aligned} P_\uparrow &:= e^\beta_\alpha \otimes \mathbb{1}_d \otimes e^\alpha_\beta \otimes \mathbb{1}_d \\ P_\downarrow &:= \mathbb{1}_d \otimes e^\beta_\alpha \otimes \mathbb{1}_d \otimes e^\alpha_\beta \end{aligned}. \tag{4.59}$$

Thus, by untwisting by way of transposition, these new operators satisfy (3.3). Commutation guarantees that their product similarly satisfies the Yang-Baxter equation. Due to the regularity condition i.e. the R-matrix with $\lambda = 0$ is the identity, we know that the derivative of this $\check{r}(\lambda)$ at $\lambda = 0$ separates into the sum of the spin up and down derivatives. This separation tells us that the Hamiltonian density as in (4.42) separates in turn. The Hamiltonian, can then be thought of as a direct sum of two commuting XX Hamiltonians of varying spins as,

$$H = \sum_{j=1}^{L} \left(Q^{\uparrow}_{j-1,j} + Q^{\downarrow}_{j-1,j} \right), \qquad Q^{\sigma} := \check{r}'_{\sigma}(0). \tag{4.60}$$

We then define the L-matrix, defined element-wise as,

$$l_{j\ \beta}^{\alpha}(\lambda) := r^{\alpha\gamma}_{\beta\delta}(\lambda) e_{j\ \gamma}^{\delta}\ . \tag{4.61}$$

This l satisfying the monodromy matrix's role in (3.6). One can further define the matrix $G(h) \in \mathrm{End}(\mathbb{C}^d \otimes \mathbb{C}^d)$ using the joint conjugation matrix $C := C^{\uparrow}C^{\downarrow}$,

$$G(h) = \exp\left(\frac{h}{2}C\right) = \cosh\left(\frac{h}{2}\right) + C\sinh\left(\frac{h}{2}\right). \tag{4.62}$$

Thus, we can use (4.61) to define another matrix,

$$L_j(\lambda) := G(h)\, l_j\, G(h). \tag{4.63}$$

The matrix $\check{r}$ is a solution to the decorated Yang-Baxter equation. One can see from the differences in the R_{12} term in the decorated versus undecorated equations that it should also obey,

$$\check{r}(\lambda+\mu)(C \otimes \mathbb{1}_{d^2})(l_j(\lambda) \otimes l_j(\mu)) = (l_j(\mu) \otimes l_j(\lambda))(\mathbb{1}_{d^2} \otimes C)\,\check{r}(\lambda+\mu) \tag{4.64}$$

Using this and the fact that l generates a representation of the Yang-Baxter algebra we may, with some extensive calculation, yields an interesting fact. That is to say there are two relations yielded as corollaries of this discussion. Taking a linear combination of the two we may find that the ratio of their coefficients which combines the spectral parameter and the parameters in G in a sine and hyperbolic sine respectively. This can be interpreted as a free parameter $\frac{\alpha}{\beta} = u$ which turns out to be the coupling in the Hubbard model [4]. There is an additional free parameter here but this is merely due to the overall factor available to the R-matrices due to the homogeneity of their defining relations. One of these relations shows us very naturally that the L matrices defined above constitute a representation of the Yang-Baxter algebra with R-matrix,

$$\check{R}(\lambda,\mu) = \beta\left(G(l) \otimes G(h)\right)\left[\check{r}(\lambda-\mu) + \frac{\alpha}{\beta}\check{r}(\lambda+\mu)\,(C \otimes \mathbb{1}_{d^2})\right]\left(G(-h) \otimes G(-l)\right). \tag{4.65}$$

This is Shastry's R-matrix. Setting $\mu = 0$ then we can obtain the Hubbard model's Hamiltonian from this, up to a twist of boundary conditions and Jordan-Wigner transformation.

4.3.2. The Hubbard Model as a Graded Model

In a similar vein to how we construct the concept of Grassman numbers to intuitively build in the fermionic anti-commuting nature to a number system, we want to find an analogous object for the local projection operators $e_{j\ \alpha}^{\beta}$ for fermions. This is done by introducing the notion of **grading** [36]. Graded vector spaces are those which are equipped with a notion of parity. More formally we may define it as follows.

Definition 18 (Graded Vector Spaces)**.** *A graded vector space is a finite dimensional vector space V, interpreted as a local state space, on which we may impose parity. Imposing parity on the entire vector space will decompose $V = V_0 \oplus V_1$ with dimensions m and n respectively. This is done via a parity map,*

$$\begin{aligned} p : V &\to \mathbb{Z}_2 \\ v &\mapsto i, \quad v \in V_i \end{aligned} . \tag{4.66}$$

For the basis $\{e_1, e_2, \ldots e_{n+m}\}$ we define the shorthand $p(\alpha) := p(e_\alpha)$.

The reintroduction of operators as Fermi operators, that is to say with a notion of parity, then requires us to construct an algebra of commuting and anti-commuting objects. Firstly, we need to extend parity to the space of linear operators on our space $\mathrm{End}(V)$. Let us choose an element $e_\alpha^\beta \in \mathrm{End}(V)$ such that $e_\alpha^\beta e_\gamma = \delta_\gamma^\beta e_\alpha$ i.e. a canonical basis element of the linear operator space. We can then naturally extend our parity map to the space through identifying $p(e_\alpha^\beta) = p(\alpha) + p(\beta)$. Indeed, this language also naturally describes products of matrices too. Since the basis elements' upper and lower indices contract when being multiplied then their contribution to the parity is doubled and hence, modulo 2, is null. Naturally, once more we may extend this to tensors of arbitrary rank. For instance, take the space $\mathcal{H} = (\mathrm{End}(V))^{\otimes L}$. Then, by the extended definition,

$$p\left(\bigotimes_{k=1}^{L} e_{\alpha_k}^{\beta_k}\right) = \sum_{k=1}^{L} \Big(p(\alpha_k) + p(\beta_k)\Big), \tag{4.67}$$

we may characterise an element $A \in \mathcal{H}$ by the equation,

$$(-1)^{\sum_{k=1}^{L}\big(p(\alpha_k)+p(\beta_k)\big)} A_{\beta_1\ldots\beta_L}^{\alpha_1\ldots\alpha_L} = (-1)^{p(A)} A_{\beta_1\ldots\beta_L}^{\alpha_1\ldots\alpha_L}. \tag{4.68}$$

To be clear then we define the parity of the object $p(A)$ as the sum of the parities of its basis indices in the obvious manner.

The next and final step in our quest to building an algebra is to define a bracket similar to the Lie bracket. Since we should allow for the existence of bosonic, commuting, and fermionic, anti-commuting, objects we must be able to treat them with the commutator and anti-commutator respectively.

Definition 19 (Super Bracket)**.** *Let $X, Y \in \mathrm{End}(V)$, then we define the super bracket $[\cdot,\cdot]_\pm$ as follows,*

$$[X, Y]_\pm = XY - (-1)^{p(X)\,p(Y)}\, YX. \tag{4.69}$$

Definition 20 (Lie-super Algebra)**.** *The space of endomorphisms on a finite dimensional vector space V may be endowed with grading. This grading allows for the decomposition $V = V_0 \oplus V_1$ with $m = \dim V_0$ and $n = \dim V_1$. Hence, $\mathrm{End}(V)$ is equipped with a super bracket and is said to be the Lie super-algebra $\mathfrak{gl}(m\,|\,n)$.*

With our algebra constructed on the single site, we now further our construction by identifying one copy as a local space embedded in a higher dimensional state space. To do so, we reintroduce local projection operators with grading.

Definition 21 (Graded Local Projection Operators)**.** *Define the matrices [35],*

$$e_{j\ \alpha}^{\beta} := (-1)^{(p(\alpha)+p(\beta))\sum_{k=j+1} p(\gamma_k)}\, \mathbb{1}_{m+n}^{\otimes(j-1)} \otimes e_{\alpha}^{\beta} \otimes e_{\gamma_{j+1}}^{\gamma_{j+1}} \otimes \cdots \otimes e_{\gamma_L}^{\gamma_L}. \tag{4.70}$$

We should note that by convention there is summation over double the double tensor indices. The index j is referred to as the site index. The local projection operator maintains the parity $p(e_{j\ \alpha}^{\beta}) = p(\alpha) + p(\beta)$.

Following (4.70) we may evaluate the same-site and split-site multiplication of two graded projection operators. Compiling these two expressions [4] we may evaluate the super bracket of two operators,

$$[e_{j\ \alpha}^{\beta}, e_{k\ \gamma}^{\delta}]_{\pm} = \delta_{jk}\Big(\delta_{\gamma}^{\beta} e_{j\ \alpha}^{\delta} - (-1)^{(p(\alpha+p(\beta))(p(\gamma)+p(\beta))}\delta_{\alpha}^{\beta} e_{j\ \gamma}^{\beta}\Big). \tag{4.71}$$

Thus, for $j = k$, giving us the structure constants of the Lie super algebra. The form of (4.70) may look familiar to the reader and shares similarity to the (4.50). This is precisely the utility of the graded vector space description. Indeed, identifying $e_1^2 = \sigma^+$, $e_2^1 = \sigma^-$, and $e_1^1 - e_2^2 = \sigma^z$ for an $m = n = 1$ space we find precisely,

$$c_j = \mathbb{1}_2^{\otimes(j-1} \otimes \sigma^+ \otimes (\sigma^z)^{\otimes(L-j)}, \tag{4.72}$$
$$c_j^\dagger = \mathbb{1}_2^{\otimes(j-1} \otimes \sigma^- \otimes (\sigma^z)^{\otimes(L-j)}, \tag{4.73}$$

as expected.

Definition 22 (Super Tensor Product)**.** *Consider the set, $\mathcal{A}$, of $m + n$ dimensional matrices with entries in $\mathcal{H}$ such that for every $X \in \mathcal{A}$ we have $p(X_{\alpha}^{\beta}) = p(\alpha) + p(\beta)$ for $\alpha, \beta = 1, \ldots, m + n$. The earlier mentioned matrix multiplication property defines $\mathcal{A}$ as an associative algebra. For any $A, B \in \mathcal{A}$ we define their super tensor product as,*

$$(A \otimes_s B)_{\beta\delta}^{\alpha\gamma} := (-1)^{(p(\alpha)+p(\beta))p(\gamma)} A_{\beta}^{\alpha} B_{\delta}^{\gamma}. \tag{4.74}$$

Definition 23 (Super Trace)**.** *Let $\mathcal{A}$ be the same as above. Suppose we have an element $A \in \mathcal{A}$. Then we define the super trace of A as,*

$$\mathrm{str}(A) = (-1)^{p(\alpha)} A_{\alpha}^{\alpha}. \tag{4.75}$$

We conclude our mathematical discussion by noting an implication of (4.74). If two elements in the defining algebra $A, B \in \mathcal{A}$ super commute, that is $[A, B]_{\pm} = 0$, then a product of their super tensor product with another two elements $X, Y \in \mathcal{A}$ will factorise. Put plainly,

$$(X \otimes_s A)(B \otimes_s Y) = XA \otimes_s BY. \tag{4.76}$$

4.3.3. Fundamental graded models

The definition of a fundamental *graded* model is a relatively simply extrapolation of the fundamental model as defined above. They are simply graded representation, that is to say for a given grading we associate a fundamental model with each solution of the Yang-Baxter equations. Except we need to add in a consistency condition alongside this [37],

$$R^{\alpha\beta}_{\gamma\delta}(\lambda,\mu) = (-1)^{p(\alpha)+p(\beta)+p(\gamma)+p(\delta)} R^{\alpha\beta}_{\gamma\delta}(\lambda,\mu). \tag{4.77}$$

The eager reader may spot that when the aggregate parity of the R-matrix is odd. That is to say if $p(\alpha)+p(\beta)+p(\gamma)+p(\delta) = 1 \mod 2$ then that element of the R-matrix vanishes. Furthermore, we define the graded L-matrix with site index j as,

$$\mathcal{L}^{\alpha}_{j\ \beta}(\lambda,\mu) := (-1)^{p(\alpha)p(\gamma)} R^{\alpha\beta}_{\gamma\delta}(\lambda,\mu) e^{\delta}_{j\ \gamma}. \tag{4.78}$$

This definition imbues the L-matrix with various properties. Notably, its behaviour under the parity map is as expected, it super commutes with its pair on any other site, and finally it satisfies an analogous (3.6)-esque relation

$$\check{R}(\lambda,\mu)\left(\mathcal{L}_j(\lambda,\nu)\otimes\mathcal{L}_j(\mu,\nu)\right) = \left(\mathcal{L}_j\otimes\mathcal{L}_j\right)\check{R}(\lambda,\mu). \tag{4.79}$$

Although we should note here that the R-matrix in this does not undergo the grading procedure. Physically we can understand this by the R-matrix's role as a scattering matrix swapping auxiliary, i.e. non-physical, spaces. Hence, as in the non-graded case $\check{R}^{\alpha\beta}_{\gamma\delta} := R^{\beta\alpha}_{\gamma\delta}$. We call this $\mathcal{L}_j$ the fundamental graded representation of the graded Yang-Baxter algebra with R-matrix $\check{R}$. To swap physical spaces we need a fermionic R-operator instead of the usual R-matrix.

Definition 24 (Fermionic R operator)**.** *We define the operator,*

$$\mathcal{R}^f_{jk}(\lambda,\mu) := (-1)^{p(\gamma)+p(\alpha)(p(\beta)+p(\gamma))} R^{\alpha\beta}_{\gamma\delta}(\lambda,\mu) e^{\gamma}_{j\ \alpha} e^{\delta}_{k\ \beta}, \tag{4.80}$$

as the fermionic R operator. By definition it has parity null We have constructed it such that it satisfies,

$$\mathcal{R}^f_{jk}(v_j,v_k)\mathcal{L}_k(\lambda,v_k)\mathcal{L}_j(\lambda,v_j) = \mathcal{L}_j(\lambda,v_j)\mathcal{L}_k(\lambda,v_k)\mathcal{R}^f_{jk}(v_j,v_k). \tag{4.81}$$

Furthermore, it satisfies the original, non-braiding, Yang-Baxter equation. Finally, if an R-matrix is regular then its fermionic cousin is a graded permutation operator and if an R-matrix is unitary then its fermionic version is yet again unitary.

We define the monodromy matrix in the same fashion,

$$\mathcal{T}(\lambda) := \mathcal{L}_L(\lambda,v_L)\cdots\mathcal{L}_1(\lambda,v_1). \tag{4.82}$$

$\mathcal{T}$ is too a representation of the graded Yang-Baxter. This can be easily seen by repeated application of the fundamental representation relations, together with the super commutation of $\mathcal{L}_j$ and $\mathcal{L}_k$ for $j \neq k$ and factorisation of the super tensor product. Expectedly, we may define the transfer matrix in precisely the same way $t(\lambda) := \mathrm{str}(\mathcal{T}(\lambda))$ with precisely the same commutation property. Likewise, the Hamiltonian density is defined analogously to the non-graded case.

We can now define a general fermionic operator X_j in a physical sense. Given a set of spinless fermions on a ring of L lattice sites following the usual anti-commutation relations then we express the general operator as,

$$X_j := \begin{pmatrix} 1-n_j & c_j \\ c_j^\dagger & n_j \end{pmatrix}. \tag{4.83}$$

Interestingly, the usual relation for the local projection operators at the same site still holds here. That is, $X_{j\ \alpha}^{\beta}X_{j\ \gamma}^{\delta} = \delta_{\gamma}^{\beta}X_{j\ \alpha}^{\delta}$. Therefore, we can replace the local projection operators with these objects. While we're at it, we might as well add anther layer of variation by assigning spin species too. In full, we then have $X_j = X_j^{\uparrow} \otimes_s X_j^{\downarrow}$. We can then also rewrite the fermionic creation and annihilation operators per spin in terms of these general operators too. Identification of these operators as the original local projection operators provides us again with the Jordan-Wigner transformation.

We now endeavour to show that the Hubbard model is, in fact, a fundamental graded model. Thus justifying the previous exposition and allowing us to utilise this built framework. Let us choose the grading $p(1) = p(4) = 0$ and $p(2) = p(3) = 1$. We take a slightly transformed version of (4.53) as our R-matrix which allows it to be compatible with this grading choice. Doing so, coupled with the replacement of non-graded with Fermi operators, allows us to find the Hamiltonian density,

$$H_{j-1,j} = -c_{j-1,a}^{\dagger}c_{j,a} - c_{j,a}^{\dagger}c_{j-1,a} + \frac{u}{2}\Big((1-2n_{j-1,\uparrow})(1-2n_{j-1,\downarrow}) + (1-2n_{j,\uparrow})(1-2n_{j,\downarrow})\Big), \tag{4.84}$$

in the usual operator expansion fashion. One can show with some algebra that is equivalent to (4.42).

Now that we have shown that this description may match the Hubbard model. With the Hamiltonian in hand, we next consider its symmetries. A crucial point here is the difference between the fermionic and non-fermionic R-matrices in the context of symmetries. Since it is the former that determines the scattering process between two physical particles it is that object which describes symmetries and has matrix elements,

$$\tilde{R}_{\beta\delta}^{\alpha\gamma}(\lambda,\mu) = (-1)^{p(\alpha)p(\gamma)}R_{\beta\delta}^{\alpha\gamma}(\lambda,\mu). \tag{4.85}$$

Although it is not immediately clear from the algebraic form, indeed the explicit form is needed for the discussion [4], we may construct two $\mathfrak{su}(2)$ symmetries from this R-matrix. To begin we define the notation,

$$\Sigma_s^{\pm} := \sigma^{\pm}\otimes\sigma^{\mp}, \qquad \Sigma_s^{z} := \frac{1}{2}\Big(\sigma^z\otimes\mathbb{1}_2 - \mathbb{1}\otimes\sigma^z\Big), \tag{4.86}$$

$$\Sigma_\eta^{\pm} := \sigma^{\pm}\otimes\sigma^{\pm}, \qquad \Sigma_s^{\eta} := \frac{1}{2}\Big(\sigma^z\otimes\mathbb{1}_2 + \mathbb{1}\otimes\sigma^z\Big). \tag{4.87}$$

These objects define the generators for two separate $\mathfrak{su}(2)$ algebras. We see here the re-emergence of spin, s, and η-spin, η, as we had previously. The spin symmetry is seen with relative ease. Indeed, defining the total spin operator,

$$S^{\alpha} := \frac{1}{2}\sum_{j=1}^{L}\sum_{a} c_{j,a}^{\dagger}(\sigma^{\alpha})_b^a c_{j,b}, \tag{4.88}$$

we may express it via the local projection operators. Doing so we see that this is related to the generator Σ_s^{α} at the root. Then by a simple calculation stemming from the definition of the graded monodromy matrix [4] and taking the super trace we arrive at,

$$\mathrm{str}[\mathcal{T}(\lambda), S^{\alpha}] = 0. \tag{4.89}$$

Therefore the total spin is conserved. Unfortunately, the η-spin does not prove conserved in such a nice manner. Instead, we look to *anti*-commutation relation between $\tilde{R}$ and the object $\Sigma_\eta^z\otimes\mathbb{1}_4 - \mathbb{1}_4\otimes\Sigma_\eta^z$. This being zero, which must be calculated explicitly, tells us that the commutation relation between the same, except for the relative minus sign in the latter term being changed to addition, is null. Eventually, by super tracing the generator out, we arrive at the conclusion that the transfer matrix anti-commutes with $\eta^{\pm}$. Setting $\lambda = 0$ one can show that the R-matrices becomes mere transposition operators and so the super trace merely yields the shift operator. The fact that $\eta^{\pm}$ anti-commutes with both the shift operator U and the transfer matrix means that it will commute with the generating function and hence the Hamiltonian - as well as the higher conserved charges - are all invariant under the action of the η algebra. So the η-spin is indeed conserved.

After a lengthy discussion about the constructing of the Hubbard model from this point of view, we may finally see if we can solve the model in this framework. First and foremost, we again require the previously mentioned upper triangular action of the monodromy matrix on the pseudo-vacuum. Utilisation of the generator's commutation relations with the transfer matrix allows us to see precisely the commutation of the monodromy matrix elements. That is to say, how the action of the monodromy matrix affects the number of electrons in the two spin species. Doing so we can assign a creation and annihilation operator to each of the spin species. With a relatively paper-hungry calculation, we involving the calculation of the graded L-matrix and all 16 elements involving Fermi operators, we arrive at the blissfully simple result,

$$t(\lambda)\left|\Omega\right\rangle = \left(\omega_1^L(\lambda) - 2\omega_2^L(\lambda) + \omega_3^L(\lambda)\right)\left|\Omega\right\rangle, \tag{4.90}$$

$$\omega_1(\lambda) = e^{2h}, \quad \omega_2(\lambda) = -\tan(\lambda), \quad \tan^2(\lambda). \tag{4.91}$$

One can now build up the one, two, and N particle cases by following the calculation in [4]. More accurately, the calculation is performed for $N = 1$ and 2 and then guessed for the variable case. Still, this is an expensive task. With this in mind, we will cite the result entirely and discuss it instead. We may find the eigenvalue of the generating function,

$$\Lambda(\lambda) = \Big[\omega_1^L(\lambda)\prod_{j=1}^{N}\frac{1}{a(\lambda,\lambda_j)} - \omega_2^L(\lambda)\prod_{j=1}^{N}\frac{1}{a(\lambda,\lambda_j)}\prod_{k=1}^{M}\frac{1}{a(\mu_k,\lambda)} - \omega_2^L(\lambda)\prod_{k=1}^{M}\frac{1}{a(\lambda,\mu_k)} + \omega_3^L(\lambda)\Big](-1)^N\prod_{j=1}^{N}\frac{\rho_{10}(\lambda,\lambda_j)}{\rho_8(\lambda,\lambda_j)}, \tag{4.92}$$

where the spectral parameters λ_j and μ_k satisfy the Bethe equations,

$$\left(\frac{\omega_1(\lambda_j)}{\omega_2(\lambda_j)}\right)^L = \prod_{k=1}^{M}\frac{1}{a(\mu_k,\lambda_j)}, \quad j = 1, 2, \ldots, N, \tag{4.93}$$

$$\prod_{j=1}^{N}\frac{1}{a(\mu_k,\lambda_j)} = \prod_{\substack{l=1\\ l\neq k}}^{M}\frac{a(\mu_l,\mu_k)}{a(\mu_k,\mu_l)}, \quad k = 1, 2, \ldots, M. \tag{4.94}$$

If we wisely choose our values of charge momenta k_j and spin rapidity Λ_k we can now clearly see how the algebraic Bethe ansatz yields our spin-charge decoupling and provides us with the (4.29). With $\Lambda(\lambda)$ in hand, we can find our shift and energy [9] eigenvalues trivially,

$$\omega(0) = \Lambda(0) = e^{i\sum_j k_j}, \qquad E = \frac{\Lambda'(0)}{\Lambda(0)} = -2\sum_{j=1}^{N}\cos(k_j) + 2u(L-N). \tag{4.95}$$

With this triumph we, at last, conclude our discussion of the Hubbard model.

[9] Technically this eigenvalue is for the Hamiltonian we derived from our graded fundamental model discussion, rather than our original (4.42).

5. Future Work

In this, our penultimate chapter, we layout some aspects of work to be carried out in the future. There is much to cover from the point of view of the Hubbard model. A discussion of bound states - in particular the string hypothesis [2], [4] - is fully missing from this work alongside the relevant thermodynamic picture. The goal in mind for this is to understand the precise fashion in which the excitations found in the Hubbard model kills the chances of finding superconductivity. Indeed, there is plenty to learn. However, we will spend this time to discuss the previously alluded to conjecture.

5.1. Conjecture on the nature of the quantum Hall effect on $\mathbb{T}^2$ in the Lowest Landau Level

The quantum Hall effect is one of the most beautiful arenas available to the condensed matter theorist. It intertwines the mathematics of topology with, what is ultimately, very simple physics. The system is simple, suppose we have a quasi two dimensional metal in which the electrons moving throughout form a Fermi liquid which is effectively constrained to the two dimensional plane. Through this plane we pass a uniform magnetic field. In the classical picture, assuming no electron-electron interactions for a moment, we see the coordinate movement of electrons in individual cycles as in Fig. 5.1. The phase space which the classical system lives on is clearly four dimensional. One can canonically transform their coordinates into a basis where these decomposes into a relative and centre of mass motion part respectively. This decomposition is important both here and in the quantum case.

One may gain interesting physics from the classical system alone when a voltage difference is placed across the plane. However, it is far more fruitful to treat the system quantum mechanically. It is a greatly unfortunate fact that we do not have the space to fully discuss its beauty here. But the interested reader should be directed to [38].

For the moment, we must stay focused on one particular aspect. That being the duality between the Calogero family of integrable systems and the quantum Hall effect. But to do so we must first understand what the Lowest Landau Level is and what utility it serves. Upon the basic quantum

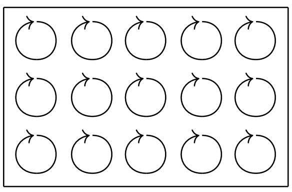

Figure 5.1. We see here the classical picture of the Hall effect. That is to say, a set of cyclically moving electric charges. One pictures here, a uniform magnetic field passing through the plane. This classical picture is equivalent to that which is seen in the coherent state basis.

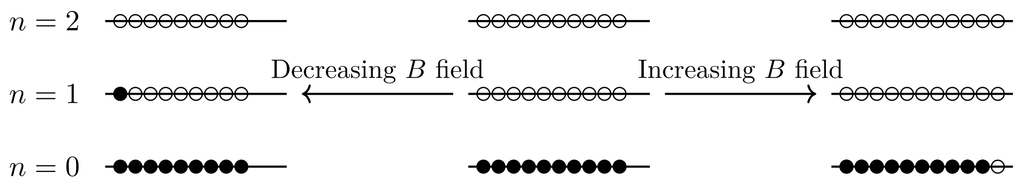

Figure 5.2. A diagram loosely showcasing the effect of a varying magnetic field i.e. deformation parameter, on the quantum Hall system. We see that when increasing the magnetic field, the particles are localised to the lowest band. Increasing the magnetic field has the effect of widening the Landau level and adding in another electron state which is not filled.

mechanical treatment of the system, one may perform a Bogoliubov transformation into new creation and annihilation operators. This transformation plays the role of the canonical in the classical case. These two new pairs mirror the relative and centre of mass motion. The energy spectrum of the system is, due to the translational invariance, highly degenerate. This degeneracy leads us to denote an infinite set of bands equidistant from one another - since the system is described in the same manner as the harmonic oscillator - as Landau Levels. This spacing is determined, up to constants, by the strength of the magnetic field. In the case of a very high magnetic field, these spacings become sufficiently large such that we can assume that all the electrons exist in the Lowest Landau Level. This projection removes half of the degrees of freedom in the system thereby reducing it an effectively one dimensional system. It was found [3] that when the quantum Hall system was placed on a disk, this projection would result in the same spectrum and solutions as the rational CMS model. Placing the system on a cylinder yielded the very same but for the cylindrical CMS model. Finally, Fourier transforming this last system led to the hyperbolic CMS model. Here we conjecture and hence lay out the evidence towards a fourth duality.

From [3], it seems that one must first choose a wise gauge and then project to the Lowest Landau Level for success. In the Lowest Landau Level, the phase space is reduced from $(\Pi_x, \Pi_y, X, Y) \mapsto (X, Y)$, which describe the guiding centre and momentum. We must then diagonalise the operator X, i.e. $X\,|s\rangle = s\,|s\rangle$. This parameter s is now our free one dimensional parameter. Finding these states $|s\rangle$ and performing the decomposition of the Lowest Landau Level states onto the multi-particle state s-state, $\langle s_1 \dots s_N|\Psi\rangle$ yields precisely the same solution as the CMS model. We now conjecture that placing the quantum Hall effect on a toroidal geometry and performing the same procedure will result in the elliptical CMS model.

Starting from the toroidal quantum Hall, we choose the τ gauge as in [39]. We can tile the torus with varying parallelograms as in Fig. 5.3. Using these parameters we define,

$$\tau := \underbrace{\frac{L_\Delta}{L_x}}_{\tau_1} + i \underbrace{\frac{L_y}{L_x}}_{\tau_2} . \tag{5.1}$$

Then we can map to the complex plane and define one of two parametrisations,

$$z = \tilde{x} + i\tilde{y},\ (\tilde{x}, \tilde{y}) \in [0, L_x) \times [0, L_y), \quad \text{or} \quad z = L_x(x + \tau y)\ (x, y) \in [0, 1)^2. \tag{5.2}$$

In any case, we choose to use the vector potential,

$$\mathbf{A} = \frac{\tilde{y}B}{\tau_2}(\tau_2 - \tau_1). \tag{5.3}$$

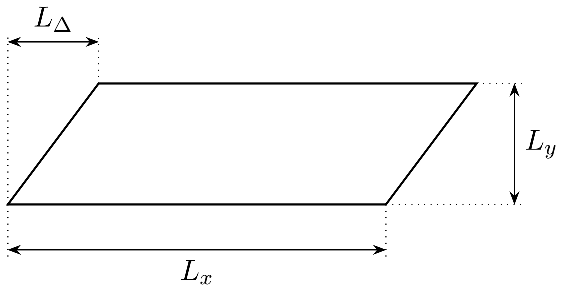

Figure 5.3. The parallelogram patches that construct the torus the parameters of which are used in defining the τ gauge.

Next, we define the magnetic translation operators, t_x and t_y. Both of which, since they are translation operators in an ultimately translationally invariant system, must commute with the Hamiltonian. Effectively, these are given as $t_x \sim e^{\partial_x}$ and $t_y \sim e^{\partial_y}$ respectively. We can diagonalise with respect to t_x and obtain a wave function for a set of particles with positions z_k [40],

$$\Psi_x(z_k) \sim \prod_{i<j} \vartheta_1 \left(\frac{z_i - z_j}{L_x} \middle| \tau \right)^q \mathcal{F}_S \left(\frac{z}{L_x}, \tau \right), \tag{5.4}$$

where z is the centre of mass position and ϑ_1 is a special case of the Jacobi theta function. The function $\mathcal{F}$ can be found in [39] and its precise form is not important here. What is vital, however, is,

$$\vartheta_1(z, \tau) \coloneqq \vartheta \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} (z, \tau). \tag{5.5}$$

As approaching from the elliptical Caloger Sutherland (eCS) model we first recall that the potential is of the form[22],

$$\wp \left(q_i - q_j \,|\, \pi, \frac{i\beta}{2} \right), \tag{5.6}$$

with $\wp$ the Weierstrauss elliptic function. The system now lives on S^1 with each particle's coordinates $q_i \in [-\pi, \pi]$. The solutions with partitions, that is to say with excitations, are given by,

$$\Psi_{\mathbf{n}}(\mathbf{q}) = \mathcal{J}_{\mathbf{N}}(\mathbf{z})\, \Psi_0(\mathbf{q}), \quad z_j \coloneqq e^{iq_j}, \tag{5.7}$$

$$\Psi_0(\mathbf{q}) \coloneqq \prod_{j<k} \theta(q_i - q_j)^\lambda. \tag{5.8}$$

Here the function $\theta(q)$ is related, up to a constant, to the function $\vartheta_1(q/2)$. The functions $\mathcal{J}$ here are the famous Jack polynomials. We now see a striking similarity between the two pictures. Indeed, we point out that these solutions are incredibly similar if we associate the quantum Hall filling fraction $\nu = 1/q$ state with the value of λ defined through the coupling $\gamma = 2\lambda(\lambda - 1)$. Furthermore, directly we see the eCS wave function is proportional up to a double product of $\vartheta_1 \left(\frac{q_i - q_j}{2} \right)$ almost precisely as in the toroidal picture.

Further research, both in the physical and mathematical domains, is required before claiming anything close to a result. But we hope that in the coming months more work can be done to prove or disprove this conjecture.

6. Conclusion

This project has traced a path from the foundations of classical integrability through to the solution of the one-dimensional Hubbard model, with the aim of providing both a mathematically rigorous and physically motivated account of quantum integrability and the Bethe Ansatz. Beginning from Hamilton's principle and the Poisson bracket structure of classical mechanics, we developed the Lax representation and the classical r-matrix, before situating these objects within the broader framework of symplectic geometry and Hamiltonian reduction. This classical groundwork was not incidental - it provided both the language and the intuition that carried through to the quantum setting.

In the quantum regime, the Yang-Baxter equation emerged as the organising principle, motivated from multiple directions and shown to underpin both the spin chain picture and the theory of factorised scattering. The Bethe Ansatz, in both its coordinate and algebraic formulations, was developed as a concrete and powerful method for solving integrable models. The Hubbard model served as the principal example throughout this discussion: its Hamiltonian was analysed in detail, its spectrum found via the Lieb-Wu equations, its hidden $SO(4)$ symmetry uncovered, and its algebraic solution constructed using the formalism of graded spaces and Shastry's R-matrix. The agreement between the coordinate and algebraic approaches is itself a satisfying consistency check on the entire framework.

Beyond the main body of the work, a conjecture was proposed regarding the quantum Hall effect on a toroidal geometry and its relationship to the elliptic Calogero-Sutherland model, extending a known duality that accounts for the disk, cylinder, and Fourier-transformed cylinder geometries. This remains, for now, a conjecture - but the evidence is suggestive and the question well-posed. It is hoped that the work laid out here provides a solid enough foundation from which to pursue it further.

A. Formalisation of Classical Integrability

A.1. Nöther's Theorem

Definition 25 (Variational Symmetry)**.** *A variational symmetry, which for our purposes is also known as a Hamiltonian symmetry, is a transformation*

$$q^i \to q^i + \varepsilon\, \delta q^i \tag{A.1}$$

where q^i are the generalised coordinates of the system such that

$$\mathcal{L} \mapsto \mathcal{L} + \varepsilon\, \frac{\mathrm{d}}{\mathrm{d}t}\Lambda, \qquad \varepsilon \ll 1. \tag{A.2}$$

Though it should be noted that this can and should be extended to include time and reveal the Hamiltonian itself as an integral of motion when $\frac{\partial}{\partial t}\mathcal{H} = 0$. It can be shown [2] that the Lagrangian derivative

$$\frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \frac{\partial \mathcal{L}}{\partial q^i}$$

transform covariantly under these variational symmetries. Hence, solutions to the Euler-Lagrange equations are themselves solutions to the Euler-Lagrange equations. It should be noted, however, that the inverse is not true. That is, there exists symmetries of the equations of motion that are not symmetries of the action. See [41] for more.

Using this we introduce Nöther's theorem.

Theorem 4 (Nöther)**.** *Consider a $2N$ dimensional dynamical system which we transform as $q^i \to q^i + \varepsilon\, \delta q^i$. Suppose this can be done for s uniquely defined transformations. For each transformation, if the system's Lagrangian varies as in $\mathcal{L} \mapsto \mathcal{L} + \varepsilon\, \frac{\mathrm{d}}{\mathrm{d}t}\Lambda$ where $\varepsilon \ll 1$ .Then there exists an independent quantity that is conserved on solutions to Hamilton's equations - or equally the Euler-Lagrange equations. These s conserved quantities are given by*

$$J_s = p_i\, \delta q^i_s - \Lambda_s \tag{A.3}$$

Proof. For each s transformations we have a δq^i_s and a Lagrangian which varies as

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial q^i}\, \varepsilon \delta q^i_s + \frac{\partial \mathcal{L}}{\partial \dot{q}^i}\, \varepsilon \delta \dot{q}^i_s = \varepsilon \frac{\mathrm{d}}{\mathrm{d}t}\Lambda_s,$$

using the product rule we see this is just

$$\frac{\mathrm{d}}{\mathrm{d}t}\Lambda_s = \frac{\partial \mathcal{L}}{\partial q^i}\, \delta q^i_s + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\delta q^i_s\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right)\delta q^i_s,$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\delta q^i_s - \Lambda_s\right) = \left[\cancelto{0 \text{ on-shell}}{\frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \frac{\partial \mathcal{L}}{\partial q^i}}\right]\delta q^i_s.$$

Implying that the following quantity is conserved in the case of each transformation

$$J_s := \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \delta q_s^i - \Lambda_s.$$

□

A.2. Liouville's Theorem

Theorem 5 (Arnold-Liouville)**.** *Let $\mathscr{P}$ be a $2N$ dimensional symplectic manifold. Suppose there exists N smooth functions defined on this manifold, $f_i \in \mathscr{F}(\mathscr{P})$ that are pairwise in involution. That is to say, they pairwise Poisson commute. Further, assuming each f_i is independent and hence their differential forms are linearly independent at each point on the manifold. Finally, we impose the existence of a level set defined by an $N-$array of constants c_i,*

$$\mathscr{P}_c = \{x \in \mathscr{P} \text{ s.t. } f_i = c_i\}. \tag{A.4}$$

Then we have the following

1. *The smooth manifold $\mathscr{P}_c$ is invariant under transformations generated by a Hamiltonian $\mathcal{H} = \mathcal{H}(f_i)$.*
2. *If $\mathscr{P}_c$ is connected and compact*[1] *then it is diffeomorphic to the N-dimensional torus $\mathbb{T}^N$ with coordinates*
$$\mathbb{T}^N = \{\Big(\varphi_1, \varphi_2, \dots, \varphi_N\Big) \mod 2\pi\}$$
3. *The motion on the level set as generated by the Hamiltonian is conditionally periodic*
$$\frac{\mathrm{d}\varphi_i}{\mathrm{d}t} = \omega_i(c)$$
The term conditionally periodic here means that for an integer valued N-dimensional vector, $\mathbf{k}$, there exists no non-zero solutions to the equation $\mathbf{k} \cdot \boldsymbol{\omega} = 0$.
4. *The EoM can be found by quadrature.*

For a thorough proof the interested reader should see [42]. However, we do not think the proof itself warrants much thought. we will outline the proof here following Arutyunov and requiring his proof of Frobenius' theorem [2]. This is of pedagogical use as the notion of toroidal motion is of importance.

Outline of proof. To begin, let's first define any necessary terms and state any theorems.

Definition 26. *An integral submanifold $\mathscr{M}$ is a connected immersed submanifold if given a distribution V the tangent space at every point in the manifold is a subspace of the distribution at that point. In the case where the two are equal it is called maximal. A distribution here effectively just means a space of directions attributed to every point on the submanifold.*

[1] It should be noted here that the result can be extended to the case where the level set is not compact if the set $\{\xi_i\}$ is complete.

Theorem 6 (Frobenius)**.** *A distribution V on a smooth manifold $\mathscr{M}$ generated by a set of m vector fields is integrable if and only if the set of vector fields is in pairwise involution.*

Owing to the pairwise involution, we have that for all pairs i, j the statement $\xi_i f_j = 0$ implies that the vector fields are all tangent to $\mathscr{P}_c$. Furthermore, the fact that the vector fields are in involution implies by Thm. 6 that $\mathscr{P}_c$ is a maximal integral submanifold for the distribution spanned by ξ_i.

Let $\{g_i^{t_i}\}$ be the set of one-parameter subgroups consisting each of the group of symplectomorphisms generated by ξ_i respectively with $t_i \in \mathbb{R}$. We can define a natural group action on the level set as

$$g^t(x) = \prod_{i=1}^{N} g_i^{t_i}(x).$$

Since ξ_i span the level set this action is transitive. We can then say that $\mathscr{P}_c \simeq \mathbb{R}^N/\Gamma$ where Γ is the isotropy group. The linear independence of the set $\{\xi_i\}$ implies that for small t where the action on the level set is effectively a sum of these vector fields then the only value of t for which the resultant action is trivial is $t = 0$. Therefore, the isotropy group is discrete and hence isomorphic to the lattice $\mathbb{Z}^N$ finally implying that

$$\mathscr{P}_c \simeq \mathbb{R}^N/\mathbb{Z}^N \simeq \mathbb{T}^N. \tag{A.5}$$

Showing that the level set is a torus is tantamount to showing that the space is parametrised by some coordinates

$$\{\Big(\varphi_1, \varphi_2, \ldots, \varphi_N\Big) \mod 2\pi\} \tag{A.6}$$

with frequencies

$$\frac{\mathrm{d}\varphi_i}{\mathrm{d}t} = \omega_i(c). \tag{A.7}$$

This formalism yields the enticing form of minute motion of the level set as the form

$$\varphi_i(t) = \varphi_i(0) + \omega_i t. \tag{A.8}$$

The final aspect of the proof we need to show is that this description yields the ability to solve the EoM by quadrature. This is quite the task. We will develop a new framework for solving classical mechanical problems to prove it.

In general, the variables f_i and φ_j do not constitute canonical coordinates i.e. in general $\{f_i, \varphi_j\} \neq \pm\delta_{ij}$. So we endeavour to build canonical coordinates using only the resources we have and points we've proved. For ease, let's assume $\mathscr{P} = \mathbb{R}^{2N}$. According to earlier points in Thm. 5, the motion occurs on an N dimensional level set of N functions we call *integrals of motion* or just *integrals*. We define γ_j with $j = 1, 2, \ldots, N$ as the fundamental cycles of the torus depending on the values of the integrals, c_i. The integrals of motion have given

$$f_i(p, q) = c_i, \quad \text{we invert to get} \quad p_i = p_i(c, q).$$

We can now define the so-called action variables, named for its similarity to the first term in the action as seen in (2.1).

$$I_i(c) = \oint_{\gamma_i} \underbrace{p_j\, dq^j}_{\alpha}, \tag{A.9}$$

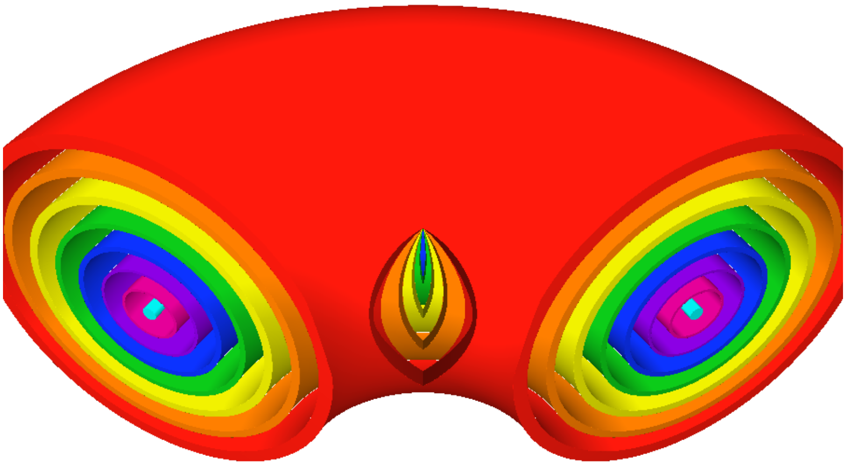

Figure A.1. A qualitative example of a set of tori to which a phase space is diffeomorphic to as in Thm 5. In particular, we choose to illustrate eight example level sets differing by their colour. These various level sets are seen to be concentric and contained within each other. Each of which corresponds to a different phase space $\mathscr{P}_i$ the system may live on.

where α is the canonical one-form. Taking these action variables, we now define angle variables θ_j by enforcing that (I_i, θ_j) form a canonical pair. Let's go about this by constructing the canonical transformation[2]

$$(p, q) \to (I, \theta).$$

We use, as a generating function, Hamilton's principal function

$$S = \int_{q_0}^{q} \mathrm{d}\tilde{q}\, p(I, \tilde{q}).$$

Then we have,

$$p_i = \frac{\partial S}{\partial q^i}. \tag{A.10}$$

Define now the angle variables

$$\theta^i := \frac{\partial S}{\partial I_i}. \tag{A.11}$$

Differentiating S, then, we get

$$\mathrm{d}S = \frac{\partial S}{\partial I_i}\,\mathrm{d}I_i + \frac{\partial S}{\partial q^i}\,\mathrm{d}q^i = \theta_i\,\mathrm{d}I_i + p_i\,\mathrm{d}q^i \tag{A.12}$$

$$\mathrm{d}^2 S = 0 \implies \mathrm{d}\theta_i \wedge \mathrm{d}I_i + \mathrm{d}p_i \wedge \mathrm{d}q^i = 0. \tag{A.13}$$

So that finally,

$$\mathrm{d}p_i \wedge \mathrm{d}q^i = \mathrm{d}I_i \wedge \mathrm{d}\theta_i = \omega, \tag{A.14}$$

the canonical two-form thereby showing that (I, θ) are indeed canonical variables.

[2]It should be noted that the action and angle variables are not unique. They can be varied by a constant and function $h(I)$ respectively without changing the equations of motion.

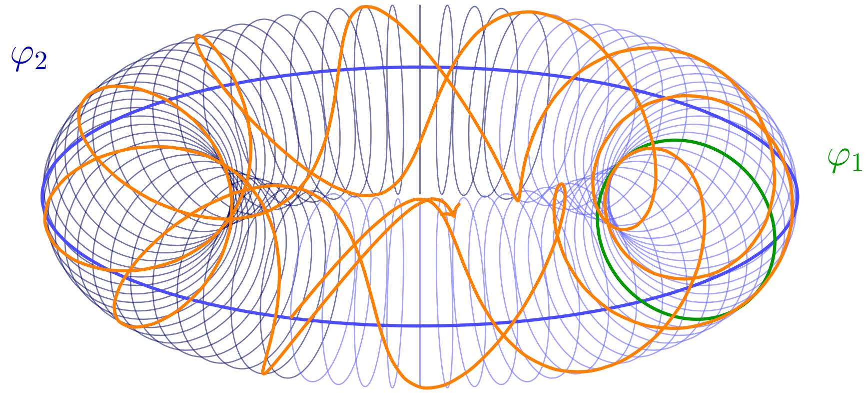

Figure A.2. A sample torus representing the reduced phase space of a system is presented here in a light blue. Furthermore, we see the trajectory taken by a particle in orange. The path taken through phase space is parametrised via two angles, φ_1 and φ_2, as in Thm. 5. These angles have associated to them two frequencies ω_1 and ω_2. It is demonstrated by the fact that the trajectory does not follow precisely the same path after a full rotation that these frequencies are not commensurable. That is to say, $\omega_1/\omega_2 \notin \mathbb{Q}$. It should be noted, however, that for a trajectory to fill the entire phase space available this must be true but even if $\omega_1/\omega_2 \in \mathbb{Q}$ the number of full rotations that need to occur in order to see the pattern may be large. Therefore, we cannot say for certainty in this case if the two frequencies are commensurable. Indeed, since they are rendered on a computer this example must be so.

Recall that the Hamiltonian $\mathcal{H} = \mathcal{H}(f_i)$ i.e. the Hamiltonian is a function of the integrals of motion. Then we can find the equations of motion - leaning on the canonical transformation - as

$$\dot{I}_i = -\frac{\partial \mathcal{H}(I)}{\partial \theta_i} = 0, \quad \dot{\theta}_i = \frac{\partial \mathcal{H}}{\partial I_i} =: \omega_i(I). \tag{A.15}$$

These can be solved easily as,

$$I_i = I_i^0, \qquad \theta_i = \theta_i^0 + \omega_i t. \tag{A.16}$$

That's it. These variables have been constructed purely from algebraic solutions to systems of equations coupled with the computation of definite integral - precisely what we mean by quadrature. □

Two frequencies are commensurable if $\frac{\omega_i}{\omega_j} \in \mathbb{Q}$. In general, the frequencies we get from a system with N constraints, ω_i, are not commensurable. Telling us that the system does not travel on a closed trajectory through the reduced phase space and is called non-degenerate. If at least two frequencies are commensurable then the motion is called degenerate. The situation where all frequencies are commensurable we call completely degenerate. It is this situation we shall focus on. Since the motion is closed, we need only one parameter to specify where we are on the phase space path. We only have one dimension and hence should expect $2N - 1$ integrals of motion. Systems in this scenario are called *superintegrable*. However, this actually implies that not all the integrals of motion are fully independent.

Claim 6. *The maximal number of independent pairwise Poisson commuting integrals of motion for a $2N$ dimensional system is N.*

Proof. Suppose there exists k integrals of motion with k corresponding independent Hamiltonian vector fields. These vector fields span a subspace $V \subset T\mathscr{P}$ at any point $x \in \mathscr{P}$. The fact there are k

integrals of motion tells us that $\dim V = k$. The skew-orthogonal compliment of V, $V^\perp$ defined as

$$V^\perp = \bigcap_{v \in V} K_v, \qquad K_v := \ker \omega(v, \cdot).$$

Since f_i are all in involution then constraining the symplectic two form to this subspace we get that $\omega|_V \equiv 0$. Therefore, $V \subset V^\perp$ and $\dim V \le \dim V^\perp$. The non-degeneracy of ω implies that the dimensionality of V and $V^\perp$ should sum to the full space $2N$. So that

$$\dim V^\perp = 2N - \dim V = 2N - k \implies k \le 2N - k$$
$$k \le N$$

as required. $\square$

In the case where the matrix $\Psi_{ij}(f) := \{f_i, f_j\}$ has rank $2(k - N)$ for a system of $2N$ dimensions with $k > N$ constraints for all c' near a c for which we investigate $\Psi(c)$, we have that $\mathscr{P}_{c'}$ is diffeomorphic to a $2N - k$ dimensional torus. The superintegrable system has a locally extended phase space diffeomorphic to a one dimensional torus i.e. S^1.

A.3. Babelon-Viallet's Theorem

We now explicitly demonstrate the necessary existence of an r-matrix in classical integrability as described in Thm. 1.

Proof. We tackle this proof in both directions separately following [43]. Assuming that the eigenvalues of L pairwise Poisson commute. Let L be diagonalised by P

$$L = P\Lambda P^{-1}.$$

Then the PB of L_1 with L_2 is given via the product rule,

$$\begin{aligned}\{L_1, L_2\} = \{P_1, P_2\}\Lambda_1 P_1^{-1}\Lambda_2 P_2^{-1} + P_2\{P_1, \Lambda_2\}\Lambda_1 P_1^{-1} P_2^{-1} - P_2\Lambda_2 P_2^{-1}\{P_1, P_2\}\Lambda_1 P_1^{-1} P_2^{-1} + \dots \\ \dots + P_1\{\Lambda_1, P_2\}P_1^{-1}\Lambda_2 P_2^{-1} + P_1 P_2 \cancelto{0}{\{\Lambda_1, \Lambda_2\}} P_1^{-1} P_2^{-1} - P_1 P_2 \Lambda_2 P_2^{-1}\{\Lambda_1, P_2\}P_1^{-1}P_2^{-1} + \dots \\ \dots - P_1\Lambda_1 P_1^{-1}\{P_1, P_2\}P_1^{-1}\Lambda_2 P_2^{-1} - P_1\Lambda_1 P_1^{-1} P_2\{P_1, \Lambda_1\}P_1^{-1}P_2^{-1} + P_1\Lambda_1 P_1^{-1} P_2 \Lambda_2 P_2^{-1}\{P_1, P_2\}P_1^{-1}P_2^{-1}.\end{aligned} \tag{A.17}$$

Where we've used the assumption that the each eigenvalue, i.e. element of Λ_i, will Poisson commute. Furthermore, define the following objects,

$$k_{12} := \{P_1, P_2\}P_1^{-1}P_2^{-1} =: -k_{21}, \qquad q_{12} := P_2\{P_1, \Lambda_2\}P_1^{-1}P_2^{-1}, \qquad q_{21} := P_1\{P_2, \Lambda_1\}P_1^{-1}P_2^{-1}. \tag{A.18}$$

With this we can rewrite the PB in a concise form by inserting $1 = P_i P_i^{-1}$,

$$\{L_1, L_2\} = k_{12}L_1L_2 + q_{12}L_1 - L_2k_{12}L_1 - q_{21}L_2 + L_2q_{21} - L_1k_{12}L_2 - L_1q_{12} + L_1L_2k_{12}. \tag{A.19}$$

Which we can rearrange using the commutator and acquire,

$$\{L_1, L_2\} = \frac{1}{2}[[k_{12}, L_2], L_1] - \frac{1}{2}[[k_{21}, L_1], L_2] + [q_{12}, L_1] - [q_{21}, L_2]. \tag{A.20}$$

Thereby, we can define the r matrix as,

$$r_{12} \coloneqq \frac{1}{2}[k_{12}, L_2] + q_{12}, \qquad r_{21} \coloneqq \frac{1}{2}[k_{21}, L_1] + q_{21}, \tag{A.21}$$

and retrieve the formula

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2]. \tag{A.22}$$

In the other direction, suppose we have the equation

$$\{L_1, L_2\} = [r_{12}, L_1] - [r_{21}, L_2],$$

then by repeated applications of the Leibniz rule we can write

$$\{L_1^n, L_2^m\} = [A, L_1] - [B, L_2], \tag{A.23}$$

for some complicated pair A, B. Taking the trace of both sides and recalling that any trace over the commutators is null, we get that

$$\{\operatorname{Tr} L_1^n, \operatorname{Tr} L_2^m\} = 0.$$

Evaluating these traces we find

$$\sum_{kl} nm\, \lambda_k^{n-1} \lambda_l^{m-1} \{\lambda_k, \lambda_l\} = 0.$$

Since we're free to choose n and m as we please, this is equivalent to the eigenvalues being in involution i.e.

$$\operatorname{Tr} L^n = 0 \iff \{\lambda_k, \lambda_l\} = 0.$$

Thus the statement is true in both directions. □

B. A Brief Soirée into Constrained Hamiltonian Systems

The purpose of this field is to describe the emergent dynamics of a system evolving under a given Hamiltonian subject to a set of constraints $\phi_i(q,p) = 0$[1]. These constraints define the constraint surface where they are all satisfied. Indeed, the notion of being on or off the constraint surface is of paramount importance. To this end we define the following notions of strong and weak equality

$$\underbrace{f = g}_{\text{strongly equal}} \qquad \underbrace{f \approx g}_{\text{weakly equal}} \quad . \tag{B.1}$$

Where to be strongly equal is to be equal across phase space whereas to be weakly equal is to be equal on the constraint surface but not the whole space.

Definition 27 (First and Second Class Functions)**.** *A function $F(x)$ is considered to be first class if for all j*

$$\{F, \phi_j\} \approx 0. \tag{B.2}$$

If this is not the case, it is said to be second class. That is to say, if there exists at least one constraint such that the Poisson bracket with the function does not weakly vanish. In particular, first class functions are defined such that they are the maximal set of functions with this property. Therefore, any function that is weakly zero must be a linear combination of first class functions.

Claim 7. *The Poisson Bracket of two first class function, $F(x)$ and $G(x)$, is too a first class function.*

Proof. By definition 27 we can say that the PB of the functions F and G can be expressed as

$$\{F, \phi_j\} = f^i_j \phi_i, \qquad \{G, \phi_j\} = g^i_j \phi_i.$$

Using the Jacobi identity then,

$$\begin{aligned} \{\{F,G\},\phi_j\} &= \{F,\{G,\phi_j\}\} - \{G,\{F,\phi_j\}\} \\ &= g^i_j\{F,\phi_i\} + \{F,g^i_j\}\phi_i - f^i_j\{G,\phi_i\} - \{G,f^i_j\}\phi_i \\ &= g^i_j f^k_i \phi_k + \{F,g^i_j\}\phi_i - f^i_j g^k_i \phi_k - \{G,f^i_j\}\phi_i. \end{aligned}$$

This is a linear combination of constraint functions and is weakly zero. □

One might think of the list of important functions and consider their differences on and off the constraint surface. Principal among them is the Hamiltonian, a first class function in itself since the

[1]This is the usual format taken in the literature. However, we have a set of functions $\mu_i(q,p)$ which we choose to set to some m_i. So we define $\phi_i \coloneqq \mu_i - m_i$.

constraints are constants on the constraint surface. Then for any dynamical function F [2] we can construct its time derivative as

$$\dot{F} \approx \{\mathcal{H} + v^a \phi_a, F\}, \tag{B.3}$$

and define the total Hamiltonian

$$\mathcal{H}_T \coloneqq \mathcal{H}' + v^a \phi_a. \tag{B.4}$$

The $\mathcal{H}'$ object here is the Hamiltonian with any additions needed to ensure that the constraints are constant. We should note that the thorough construction of these points have been skipped for brevity and we recommend the reader to consult [12] for further reading. The coefficients v^a here are completely arbitrary functions. So let's consider the difference in F after some infinitesimal time evolution by δt by taking v^a and $\tilde{v}^a$. This difference is given by

$$\delta F = \frac{\partial F}{\partial t}\bigg|_{\delta t}^{v} \delta t - \frac{\partial F}{\partial t}\bigg|_{\delta t}^{\tilde{v}} \delta t = \delta t \Big(\{v^a \phi_a, F\} - \{\tilde{v}^a \phi_a, F\} \Big) = \delta v^a \{\phi_a, F\}. \tag{B.5}$$

When F is a physical quantity the arbitrariness of δv forces the function to be first class. This is an example of how first class constraints generate gauge transformations.

We now turn our attention to second class constraints. In particular, recall the matrix $\Psi_{ij} \coloneqq \{\phi_i, \phi_j\}$ defined during our discussion of $k > N$ constraints in Appendix A.2. However, we now discuss it in the case of independent or what are known as irreducible constraints.

Claim 8. *The matrix Ψ_{ij} is weakly invertible if and only if there exist no first class constraints in the set of constraints $\{\phi_i\}$.*

Proof. If $\det \Psi \approx 0$ then there exists a non-trivial solution of u^j such that

$$\Psi_{ij} u^j = 0.$$

Then define the constraint $\varphi = u^j \phi_j$. Clearly it is first class. □

Since Ψ is weakly invertible then on the constraint surface we can imagine the hamiltonian vector field ξ_i associated with $\phi_i \sim \mu_i$ to be transversal to the constraint surface. That is to say, by following motion generated by the constraints, we do not lose information. Consider now two functions F, G, we can weakly extend these using second class constraints χ_a,

$$\tilde{F} \coloneqq F + v^a \chi_a, \qquad \tilde{G} \coloneqq G + u^b \chi_b. \tag{B.6}$$

Notice that,

$$\{\tilde{F}, \chi_i\} = \{F, \chi_i\} + v^a \{\chi_a, \chi_i\} + \chi_a \{v^a, \chi_i\} \approx \{F, \chi_i\} + v^a \{\chi_i, \chi_a\},$$

which tells us that we can define the vector v^a as

$$v^a = -C^{ia} f_i, \tag{B.7}$$

[2] Not necessarily a first class function.

where,

$$C^{ia} = \left(\Psi^{-1}\right)^{ia}; \qquad f_a := \{F, \chi_a\}. \tag{B.8}$$

Therefore,

$$\{\tilde{F}, \chi_i\} \approx 0. \tag{B.9}$$

A similar vector, u^b, can be constructed for G. To progress in our discussion of integrability, we need to ensure that our notion of Poisson bracket reflects the constraint discussion. We do this by defining a new type of bracket.

Definition 28 (Dirac Bracket)**.** *Consider the following bilinear function,*

$$\{\star, *\}_D : \mathscr{F}(\mathscr{P}) \times \mathscr{F}(\mathscr{P}) \to \mathscr{F}(\mathscr{P})$$

using a previously defined Poisson bracket. This is called a Dirac bracket and is explicitly defined as,

$$\{f, g\}_D := \{f, g\} - f_a C^{ab} g_b. \tag{B.10}$$

Claim 9. *The Dirac bracket forms a valid Poisson bracket.*

Proof. For the Dirac bracket to be a valid PB as defined in 3 it must obey the following properties:

1. Bi-linearity, **2.** $\{f, f\}_D = 0$, **3.** Leibniz Identity, **4.** Jacobi Identity,

which we show here. To begin, bi-linearity is trivial from the base bracket's status as Poisson. To prove that the bracket of two identical functions vanishes consider

$$\{f, f\}_D = \cancelto{0}{\{f, f\}} - f_a f_b C^{ab}.$$

Since C is defined as Ψ^{-1} it must be anti-symmetric. Therefore the property holds. The Leibniz rule for the first term is trivial. We show the second term here,

$$\begin{aligned}\{fg, \chi_a\}C^{ab}\{hk, \chi_b\} = fh\{g, \chi_a\}C^{ab}\{k, \chi_b\} + fk\{g, \chi_a\}C^{ab}\{h, \chi_b\} + \dots \\ \dots + gh\{f, \chi_a\}C^{ab}\{k, \chi_b\} + gk\{f, \chi_a\}C^{ab}\{h, \chi_b\}.\end{aligned}$$

Finally, the proof for the Jacobi identity is computational intensive and so we point the reader to Appendix B in [2]. ☐

One of the most intuitive aspects of this definition is that the Dirac bracket on the constraint surface is precisely the Poisson bracket of our extended functions. This tells us that the Dirac bracket is a worthwhile and relevant extension. This is a simple calculation,

$$\{F, G\}_D = \{\tilde{F}, \tilde{G}\} - \{\tilde{F}, u^a \chi_b\} - \{v^a \chi_a, \tilde{G}\} + \{v^a \chi_a, u^b \chi_b\} - f_a C^{ab} g_b$$

which is weakly equal to

$$\{F, G\}_D \approx \{\tilde{F}, \tilde{G}\} - u^b\{\tilde{F}, \chi_b\} - v^a\{\chi_a, \tilde{G}\} + v^a u^b\{\chi_a, \chi_b\} - f_a C^{ab} g_b.$$

Recalling the above definition and the fact that the extended functions Poisson commute with the second class constraints we get,

$$\{F,G\}_D \approx \{\tilde{F},\tilde{G}\} - f_i C^{ia}\psi_{ab}C^{jb}g_j - f_a C^{ab}g_b$$

using the antisymmetry of C we have, $\{F,G\}_D \approx \{\tilde{F},\tilde{G}\} + f_b C^{bj}g_j - f_a C^{ab}g_b \approx \{\tilde{F},\tilde{G}\}$.

Since the Dirac bracket of any extended function with the second class constraints is weakly null by definition, it follows that all constraints will commute in this way with respect to each other. In a way this tells us that the even second class constraints become effectively first class on the constraint surface.

As we said earlier, the functions v^a are arbitrary, so one might ask what happens if we change them? Any physical quantity of course shouldn't change. Indeed, the Dirac bracket respects this property. Defining $\bar{\chi}_a = m_a{}^b\chi_b$ we get that the new Dirac bracket is given by

$$\{F,G\}_{\bar{D}} = \{F,G\} - \{F, m_a{}^{\mu}\chi_\mu\}\bar{C}^{ab}\{G, m_b{}^{\nu}\chi_\nu\}, \quad \bar{C}^{ab} = (\{\bar{\chi}_a,\bar{\chi}_b\})^{-1} = (m^{-1})^a{}_i C^{ij}(m^{-1})^b{}_j$$
$$\{F,G\}_{\bar{D}} = \{F,G\} - \{F,\chi_\mu\}m_a{}^{\mu}\cdot(m^{-1})^a{}_{\mu}C^{\mu\nu}(m^{-1})^b{}_{\nu}\cdot m_b{}^{\nu}\{G,\chi_\nu\} = \{F,G\}_D.$$

It is specifically all of this discussion about the constraint surface that provides so much use. Identifying the region of phase space laying on the constraint surface with our reduced phase space as described in Thm. 2 that allows us to describe the dynamics. Here, then, the hamiltonian vector fields are generated now using the Dirac bracket. In particular, there exists an equivalence class of these vector fields due to the arbitrariness of v^a. This procedure can also be repeated by imposing a set of conditions

$$\gamma_i(x) = 0, \quad i = 1,\ldots,\dim\mathfrak{g}_m$$

such that

$$\{\phi_a,\gamma_i\} \not\approx 0.$$

The assignment of these functions is known as gauge fixing. The Dirac bracket in this case is constructed in precisely the same fashion except for the fact that Ψ now takes both ϕ_a and γ_i functions. In any case, the evolution equation can be expressed in two equivalent ways

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{\mathcal{H},\tilde{f}\}_D = \{\mathcal{H},\tilde{f}\} \approx \{\mathcal{H}_E,f\}_D \approx \{\mathcal{H}_E,f\}, \tag{B.11}$$

where the subscript E refers to the canonical Hamiltonian with any consistency additions made and then arbitrary functions of constraints added. Again, for more detail see [12].

Thus concludes our digression into constrained Hamiltonian systems.

C. Symmetric Functions

A topic oft used knowingly or unknowingly in quantum mechanics is that of symmetric functions. In this text it is most useful in the context of symmetric polynomials, though can be generalised by taking a countably infinite number of variables to symmetric functions. This appendix serves as the home of the mathematics needed to understand our uses of symmetric polynomials throughout the text. We will discuss Young tableaux and their relationship to irreducible representations of the symmetry group as well as other consequences.

C.1. Irreducible Representations of $\mathfrak{S}_N$

Suppose we define the operator P_{12}, effectively the split Casimir in (10), which permutes $1 \leftrightarrow 2$. This can be represented as a cycle (1 2). As seen in 3.1.2, we were able to utilise conjugations of operators to represent a permutation in a set of neighbouring transpositions[1]. This can be generalised to describe one operator in terms of conjugations of others without changing the cyclic structure. This allows us to define conjugacy classes around the base cyclic structure i.e. the set of all sequences of objects that represent the same fundamental cycle.

Another way of framing the problem of conjugacy classes is that of integer partitions. Specifically, if we are looking at the group $\mathfrak{S}_N$ then we can define that partition $\lambda = \{\lambda_1, \ldots, \lambda_k\}$ with $\lambda_i \geq \lambda_{i+1}$ and $N = \sum_{i=1}^{k} \lambda_i$. We can identify this with the set of permutations with cycles of lengths $\lambda_1, \lambda_2, \ldots, \lambda_k$. It's also worth noting that two partitions that differ only by a string of zeroes are considered equivalent. The non-zero λ_i values are called parts and the number of parts is called the length $l(\lambda)$.

Now we can represent each of these conjugacy classes, or equivalently each of these partitions, by a Young diagram. The construction of a Young diagram is easy. Working with S_N, suppose you draw boxes starting from the top left with f_s boxes in row s and μ_k boxes in column k. Then as long as $f_s \leq f_{s+1}$ and $\mu_k \leq \mu_{k+1}$ with the number of total boxes summing to N - this is a valid Young diagram. This gives you a particular partition with $\lambda_i = f_i$ describing a conjugacy class with each μ_k representing a k-cycle. It's also worth noting that two partitions that differ only by a string of zeroes are considered equivalent.

Each partition λ defines a unique irreducible representation of $\mathfrak{S}_N$. If we now deal with functions on the symmetric group, we can define the left and right regular representations of the symmetric group itself,

$$\pi(\varsigma)f(\sigma) = f(\varsigma^{-1}\sigma), \qquad \pi'(\varsigma)f(\sigma) = f(\sigma\varsigma), \tag{C.1}$$

respectively. Both representations can be decomposed into a sum of irreducible representations simply

[1] Indeed, this can be done in general.

as,

$$\pi \simeq \bigoplus_{\lambda} \dim \pi_\lambda \pi_\lambda, \tag{C.2}$$

where π_λ is the irreducible representation corresponding to the partition (or equivalently Young diagram) denoted by λ.

C.2. Symmetric Polynomials

As mentioned, symmetric polynomials are ubiquitous in this realm of quantum integrability. To that end, it is vital that we have a decent understanding of their use cases. We will largely work through this understanding by exploring varying examples mixed with some theory.

Firstly, let's work with some bases of symmetric functions. The first is,

$$p_\lambda := \prod_i p_{\lambda_i}, \qquad p_{\lambda_i} = \sum_j q_k^{\lambda_i}. \tag{C.3}$$

The second is defined as the sum of all distinct monomial that can be obtained from q^λ by permutations of the individual parts,

$$m_\lambda \equiv m_{\lambda_1 \lambda_2 \ldots} := \sum_{\sigma \in \mathfrak{S}_N} \operatorname{sgn}(\sigma) q^{\lambda 1}_{\sigma(1)} q^{\lambda 2}_{\sigma(2)} \cdots q^{\lambda N}_{\sigma(N)}. \tag{C.4}$$

We note that if the number of variables then this is just null for the same reason that a differential k-form defined on an $m < k$ dimensional manifold is null.

C.2.1. Generalised Hermite Polynomials

Suppose λ is a partition and then define the monomial,

$$q^\lambda := \prod_i q_i^{\lambda_i}, \tag{C.5}$$

and the operator,

$$\mathscr{H} = \sum_i \left(-\frac{\hbar^2}{2} \partial_i^2 + \hbar \omega q_i \partial_i \right) - \hbar \gamma \sum_{i<j} \frac{1}{q_{ij}} (\partial_i - \partial_j). \tag{C.6}$$

Clearly, this operator does not preserve the space spanned my monomials since there is a reciprocal term.

Claim 10. *The operator $\mathscr{H}$ preserves the space of symmetric polynomials.*

Proof. Suppose $\lambda_i > \lambda_j$, let's consider the final term exclusively in the operator.

$$\frac{1}{q_{ij}} (\partial_i - \partial_j)(q_i^{\lambda_{\{i}} q_j^{\lambda_{j\}}}) = \lambda_i \sum_{k=0}^{\lambda_i - \lambda_j} q_i^{\lambda_i - k} q_j^{\lambda + k} - \lambda_j \sum_{k=0}^{\lambda_i - \lambda_j} q_i^{\lambda_i - k - 1} q_j^{\lambda + k - 1}. \tag{C.7}$$

Hence, with some changing of index signs it can be seen that this formula is symmetric. The remaining action of $\mathscr{H}$ is trivial. ∎

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